REPAIRING UNDESIGNED RESPONSE SURFACE EXPERIMENTS TO MINIMIZE BIAS

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SUMMARY

It is not uncommon that an experiment is decided upon, data collected, and conclusions drawn without considering the statistical properties of the experiment or its design. It would be advantageous to develop a technique to augment these undesigned experiments with additional observations that would enhance the statistical properties of the experiment. The purpose of this investigation was to develop a technique for selecting additional runs so that bias is minimized. Efforts are limited to the first and second order models.

This investigation develops an objective function that is based on making design moments equal region moments. Minimization of this function leads to the reduction of bias. The Hooke and Jeeves pattern search is used to minimize the function. The additional runs are selected sequentially.

It is concluded that an undesigned response surface experiment can be improved by the sequential addition of runs selected by minimizing the objective function. The method also indicates when no improvement in bias is possible. Simultaneous selection of the additional runs instead of sequential selection is found undesirable as the maximum reduction of bias may not be found.

CHAPTER I

INTRODUCTION

General

Often experimental data is used in experiments without consideration of statistical properties. This may arise due to a lack of knowledge or experience on the experimenters' part or from using existing data. In such cases important effects may be confounded, regression coefficients biased, and, in general, the data may be unusable for statistical inference. These are referred to as undesigned experiments.

A statistician may be tempted to rerun the experiment using an appropriate designed experiment and throw away the previous data. Besides throwing away information that was collected at a cost of time and money, this may involve a further expenditure that is above budget limitations. A more desirable approach would be to use the existing data along with additional experimental runs so as to improve the statistical properties in relationship to some criterion. This is considered "repairing" the undesigned experiment. Since budget limitations may preclude too many additional runs, the procedure should add runs so that improvement is sequential, not demanding a fixed number of runs.

If the statistician had an opportunity prior to the collection of data, he could choose one of a number of experimental designs to use based on the works of Davies (4), Fisher-Yates (7), Hicks (10), Kempthorne (12), Kiefer (14), Kennard and Stone (13), and others. He could also use the techniques of response surface methodology developed by Box, exhibited by Box and Wilson (3) and summarized in a book by Myers (16). These techniques rely on designing the experiment prior to collecting the data. Most use the method of least squares to estimate parameters, and they usually include analysis of variance.

Response Surface Methodology

Response surface methodology assumes that a response where the

$$\eta = f(\xi_1, \xi_2, \ldots \xi_k)$$

actual functional form is not known, can be approximated in some region R of the ξ space by a graduating function $g(\underline{\xi}, \underline{\theta})$, where $\underline{\theta}$ is a vector of adjustable constants. The graduating functions usually employed are low order polynomials in the variable ξ, and often a first or second order model is selected. The objective of the experiment is to estimate the vector $\underline{\theta}$ and to perform an analysis on the fitted surface. The experiment consists of setting the levels of the variables, ξ, and measuring the response a number of times.

In designed experiments the levels of the variables are selected according to some appropriate criterion. One run of the experiment can be expressed as a vector, say

$$\underline{\xi}'_u = (\xi_{1u}, \xi_{2u}, \ldots \xi_{iu} \ldots \xi_{ku}) \ , \quad u = 1,2,\ldots,N$$

where i denotes the variable and u denotes the run number. An $n \times k$ matrix, D, with rows, $\underline{\xi}'_u$, is called the design matrix. In undesigned experiments the "design" matrix has not been properly selected or data available could not be represented in a standard design form. The term, design matrix, in this study, is used to refer to the matrix, D, but not to imply a designed experiment.

Purpose and Scope

The purpose of the investigation is to develop a technique for repairing undesigned response surface experiments. Additional runs will be added to the existing undesigned experiment so as to reduce the average bias of the regression coefficients. The repairing of first order models has no application in response surface experiments where the model is used only for steepest ascent. It is included in this study though for completeness. The second order model, where the true response is third order, is the primary interest of this investigation.

CHAPTER II

LITERATURE REVIEW

Few repair techniques have been reported in the literature on experimental design. Dykstra (5) examined the use of three criteria for selecting additional runs. They were:

1) minimize regression sum of squares,
2) maximize residual sum of squares,
3) minimize multiple correlation.

He added pairs of additional runs sequentially, maintaining the average level of the independent variable so that orthogonal blocking was achieved. The results indicated that the minimum regression criterion was inferior to the minimum correlation criterion and that a procedure combining the use of the maximum residual and minimum correlation might be best.

Gaylor and Merrill (8) developed a technique for adding runs to an undesigned experiment for a first order model in order to remove the correlation among independent variables while providing minimum variance estimates of the regression coefficients. The removal of correlation among independent variables is desirable in the first order case to approach an orthogonal design. Gaylor and Merrill's region of interest is a hypercube centered about the origin bounded by hyperplanes at -1 and +1 in each direction. Points are added at the 2^p corners of this regions such that the average level of the j^{th} transformed independent variable is zero; i.e.

$$\bar{x}_j = 0 \tag{2.1}$$

for all j. This can only be approximated in general since

$$\sum_{i=1}^{N} x_{ij} \tag{2.2}$$

will not be integer for all j.

Gaylor and Merrill developed equations for determining the number of runs, n_i, to be added at the i^{th} corner out of a total number of runs, M. These are based on maximizing the determinant of $(X'X)$, where $(X'X)$ is the matrix of the corrected sum of squares and cross-products of the independent variables, subject to the condition that off-diagonal elements of $(X'X)$ are zero after adding points. Practical difficulties arise in that n_i may turn out to be noninteger and sometimes even negative. Rounding off the n_i's establishes a set of possible solutions. To choose between these solutions, the determinent of the inverse of $(X'X)$; i.e. $|(X'X)|$, for the undesigned experiment is compared to $|(X'X)^{-1}|$ after augmentation at the various solutions. The solution minimizing the augmented $|(X'X)^{-1}|$ is chosen. Minimizing $|(X'X)^{-1}|$ is the equivalent to maximizing $|(X'X)|$.

Given that only M more runs can be made, Gaylor and Merrill show that augmenting existing data was better than running a designed experiment of M runs.

Dykstra (6) sequentially adds runs in the design space by evaluating Var $(\hat{y})$ at each point from a list of candidate points, selecting as the next run the point where Var $(\hat{y})$ is the greatest. It can be shown that this approach maximizes $|(X'X)|$. The choise of candidate points is determined by the model and may present some problems in complex spaces. For the first order case, only corner points need be specified but for the second order model, axial and center points are included as well as corner points. The ratio of $|(X'X)|$ after augmentation to $|(X'X)|$ of the undesigned experiment is used to show the improvement

Hebble and Mitchell (9) illustrate Dykstra's augmentation techniques with a wider range of examples. The problem, the design criterion, and the method of design augmentation are essentially the same. Only a minor difference occurs between the methods of Dykstra and Hebble and Mitchell. Hebble and Mitchell search the entire region of interest, requiring only that the points be experimentally feasible, using grid search for the two-variables case and a random search technique for the case of three or more variables. They point out that a sequential solution; i.e., adding one point at a time, does not generally yield that set of m points which maximizes $|(X'X)|$, but that approach is used for simplicity.

Bias, caused by fitting an inadequate model, is pointed out as being a more important source of error in many cases. The use of the $(X'X)$ criterion does not consider bias as the assumption must be made that the model fits.

Another approach is taken by Wynn (17), using the "D" and "G" optimality criterion as established by Kiefer (14). Wynn adds points to the design corresponding to points of maximum variance of the least squares estimate of the response mean for the regression model. This is essentially maximizing $|(X'X)|$ as each point is added and suffers the same restriction as Dykstra (6) and Hebble and Mitchell (9) in protecting against bias. The assumption must be made that the model is correct.

A method proposed by McCaa (15) gives explicit consideration to the possibility that the model in incorrect. He uses Box and Draper's (1) development of average mean square error

$$J = \{[n/\sigma^2] \int_R E[\hat{y}(\underline{x}) - f(\underline{x})]^2 dx\}/\int_R dx \tag{2.3}$$

or

$$J = V + B \tag{2.4}$$

where for the first order case

$$V = 1 + \sum_{j=1} \{(c^{jj}/(k + 2))\} \tag{2.5}$$

and

$$B = 1/(k + 2) \sum_{g=1}^{k} \left\{ \sum_{i=1}^{k} \sum_{j=1}^{k} \alpha_{ij} \sum_{h=1}^{k} c^{gh}[hij] \right\}^2 \tag{2.6}$$

$$+ \left\{ \sum_{i=1}^{k} \sum_{j=1}^{k} \alpha_{ij} \; ([ij] = \delta_{ij}/(k + 2)) \right\}^2$$

$$+ \frac{\left[2(k+2) \sum_{i=1}^{k} \alpha_{ii}^{\;2} + (k+2) \sum_{i=1}^{k} \sum_{j=1+1}^{k} \alpha_{ij} - 2(\sum_{i=1}^{k} \alpha_{ii}) \right]^2}{(k+2)^2 (k+4)}$$

V is the variance of y integrated, or averaged, over the region R, while B is the square of the bias, integrated over the region. Average bias, B, is minimized for a given α_{ij} by setting design moments equal to region moments yielding

$$\sum_j \sum_{l \geq j} \sum_{p \geq l} (\sum_i x_{ij} x_i \, x_{ip}) = 0 \tag{2.7}$$

$$\sum_j \sum_{l \geq j}^{k} (\sum_i x_{ij} x_{ie}) = 0 \tag{2.8}$$

$$\sum_j [\sum_i (x_{ij}^{\;2}) - N/(k + 2)] = 0 \tag{2.9}$$

which can be accomplished by minimizing the function

$$F(\underline{x}) = \sum_j (\sum_i x_{ij})^2 + \sum_j (\sum_i (x_{ij}^{\;2}) - N/(k + 2))^2 \tag{2.10}$$

$$+ \sum_j \sum_{l \geq j}^{k} \sum_{p \geq l}^{k} (\sum_i x_{ij} x_{il} x_{ip})^2 + \sum_j \sum_{l \geq j} (\sum_i x_{ij} x_{il})^2$$

For a given set of points, N, F is constant. Additional points can be added then in order to minimize F. Expressing the determined portion of F as constants

$$\sum_i^N x_{ij} = C_j' \ , \tag{2.11}$$

$$\sum_i^N (x_{ij} - N/k + 2) = C_j'' \ , \tag{2.12}$$

$$\sum_{i}^{N}(x_{ij}x_{il}x_{ip}) = C_{jlp} \quad , \tag{2.13}$$

and

$$\sum_{i}^{N}(x_{ij}x_{il}) = C_{jl} \quad , \tag{2.14}$$

McCaa rewrites F as

$$F(\underline{x}) = \sum_{j}\left(C_j' + \sum_{i=n+1}^{M} x_{ij}\right)^2 + \sum_{j}\left(C_j'' + \sum_{i=n+1}^{M} x_{ij}^{\ 2}\right)^2 \tag{2.15}$$

$$+ \sum_{j}\sum_{l\geq j}^{k}\sum_{p\geq l}^{k}\left(C_{jep} + \sum_{i=n+1}^{M} x_{ij}x_{il}x_{ip}\right)^2$$

$$+ \sum_{j}\sum_{l\geq j}^{k}\left(C_{jl} + \sum_{i=n+1}^{M} x_{ij}x_{il}\right)^2$$

Using this function on ten randomly generated undesigned experiments, he concluded:

> 1) An undesigned, statistically weak response surface experiment may be significantly improved by sequentially adding new observations in the number and manner dictated by the minimization of the function $F(\underline{X})$ which yields design with J values close to the optimal values of $\underline{J}$.
>
> 2) The determinant of the $(\underline{X}'\underline{X})$ matrix criterion cannot be considered a valid design criterion in conjunction with the minimization of mean square error.
>
> 3) The assumption that minimum mean square error is dominated by average squared bias is valid only so long as the ratio of model parameters to σ is sufficiently large.

In the development of $F(\underline{x})$ the second term comes from the requirement that the pure second design moment equals the region moment

$$\sum_i x_{ij}^2 = 1/(k + 2) \tag{2.16}$$

or

$$\sum_i x_{jk}^2 = N/(k + 2) \tag{2.17}$$

As additional runs are added, N increases to N + m. This is not considered in

$$\sum_j (C_j'' + \sum_{i=n+1}^{M=N+m} x_{ij}^2)^2 \tag{2.18}$$

the second term of 2.15. This may have had a significant effect on the augmentation points selected and the conclusions reached.

This study, while primarily concerned with the second order case, will include a reformulation of $F(\underline{x})$ for the first order model and the conclusions will be reexamined.

CHAPTER III

PROCEDURE

Development of Average Bias

The goal of this investigation is to develop a technique for repairing an undesigned experiment used to fit a second order model with a limited number of points so as to minimize the average squared bias. The choice of this criterion is based on work done by Box and Draper (1,2). They show that all-bias designs more closely approach optimal designs than those designs considering variance alone, even when average variance over the region is larger than average bias. Repair techniques based upon variance alone, using the $(X'X)$ criterion, do not offer protection against bias.

In response surface methodology the response η is usually approximated by a low order polynomial

$$y = \underline{x}'\underline{\beta} + \varepsilon \tag{3.1}$$

where y is the observed response, $\underline{\beta}$ the vector of the unknown constants, $\underline{x}$ the transformed levels of the variables ξ_1, and ε the random error assumed to have expectation 0 and variance σ^2. The vector $\underline{\beta}$ is estimated by the method of least squares.

The predicted response is

$$\hat{y} = \underline{x}'\hat{\underline{\beta}} \tag{3.2}$$

and in addition

$$\text{Var } \hat{y} = \underline{x} \text{ Var}(\hat{\underline{\beta}})\underline{x} \tag{3.3}$$

where

$$\text{Var } \hat{\underline{\beta}} = E[\hat{\underline{\beta}} - \underline{\beta}][\hat{\underline{\beta}} - \underline{\beta}] \tag{3.4}$$

or

$$\text{Var } \hat{\underline{\beta}} = \sigma^2(X\ X)^{-1} \tag{3.5}$$

In the following

$$y(\underline{x}) = \underline{x}_1\underline{\beta}_1 \tag{3.6}$$

and

$$f(\underline{x}) = \underline{x}_1\underline{\beta}_1 + \underline{x}_2\underline{\beta}_2 \tag{3.7}$$

where $f(\underline{x})$ is the true response of order d_2 and $y(\underline{x})$ is the assumed model of order d_1. The elements of the vector $\underline{x}_1$ are powers and products of order d_1 or less of the variables x_i and $\underline{x}_2$ is a vector whose elements are powers and products of order $d_1 + 1,\ldots,d_2$ of the variables x_i. The vectors $\underline{\beta}_1$ and $\underline{\beta}_2$ are the coefficients of the elements in $\underline{x}_1$ and $\underline{x}_2$. The matrices X_1 and X_2 have as rows $\underline{x}_1'$ and $\underline{x}_2'$ respectively and each now represents one run of the experiment. There are a total of N runs.

Using

$$\hat{y}(\underline{x}) - f(\underline{x}) = \{\hat{y}(\underline{x}) - E\ \hat{y}(\underline{x})\} + \{E\hat{y}(\underline{x}) - f(\underline{x})\} \tag{3.8}$$

in equation 2.3 and 2.4 Box and Drapper (1) write the average variance as

$$V = \Omega \int [\hat{y}(\underline{x}) - E\hat{y}(\underline{x})]^2 d\underline{x}$$

and the average squared bias as

$$B = \Omega \int [E\hat{y}(\underline{x}) - f(\underline{x})]^2 d\underline{x} \qquad (3.10)$$

where $\Omega = 1/\int_R d\underline{x}$

Defining the moments of the design as

$$M_{11} = N^{-1}X_1'X_1 \quad , \qquad (3.11)$$

$$M_{12} = N^{-1}X_1'X_2 \quad , \qquad (3.12)$$

$$M_{22} = N^{-1}X_2'X_2 \qquad (3.13)$$

and region moments as

$$\mu_{11} = \Omega \int_R \underline{x}_1'\underline{x}_1 d\underline{x} \qquad (3.14)$$

$$\mu_{12} = \Omega \int_R \underline{x}_1'\underline{x}_2 dx \qquad (3.15)$$

$$\mu 22 = \Omega \int_R \underline{x}_2'\underline{x}_2 dx \qquad (3.16)$$

J can now be written as

$$J = \text{trace}\ [\mu_{11}M_{11}^{-1}\ ] + \underline{\alpha}_2'[(\mu_{22} - \mu_{12}'\mu_{11}^{-1}\mu_{22}) \tag{3.17}$$

$$+ (M_{11}^{-1}M_{12} - \mu_{11}^{-1}\mu_{12})\mu_{11}(M_{11}^{-1}M_{12} - \mu_{11}^{-1}\mu_{12})]\underline{\alpha}_2$$

and

$$V = \text{trace}\ [\mu_{11}M_{11}^{-1}] \tag{3.18}$$

$$B = \underline{\alpha}_2'[(\mu_{22} - \mu_{12}'\mu_{11}^{-1}\mu_{12})$$

$$+ (M_{11}^{-1}M_{12} - \mu_{11}^{-1}\mu_{12})\mu_{11}(M_{11}^{-1}M_{12} - \mu_{11}^{-1}\mu_{12})]\underline{\alpha}_2 \tag{3.19}$$

where $\underline{\alpha}_2 = \sqrt{N}\ \underline{\beta}_2/\sigma$.

J would be an excellent criterion to use for selecting additional points for repairing a design since it contains terms for both variance and bias. However, this is not possible, as the bias term, B, relies on the unknown $\underline{\alpha}_2$ vector. In Box and Draper (2) it is shown that B can be no smaller than the positive term

$$\underline{\alpha}_2'(\mu_{22} - \mu_{12}'\mu_{11}^{-1}\mu_{12})\underline{\alpha}_2$$

and that, for consideration of bias alone, a sufficient condition for minimizing B is that design moments should be equal to region moments up to and including order $d_1 + d_2$. This condition can be expressed as

$$M_{11} = \mu_{11} \tag{3.21}$$

$$M_{12} = \mu_{12} \tag{3.22}$$

which in 3.19 makes

$$M_{11}^{-1}M_{12} = \mu_{11}^{-1}\mu_{12} \tag{3.23}$$

or

$$B = \underline{\alpha}_2'(\mu_{22} - \mu_{12}'\mu_{11}^{-1}\mu_{12})\underline{\alpha}_2 \tag{3.24}$$

the minimum value of B.

The shape of the region of interest R will affect the moment matrices above. For example, with $d_1 = 2$, $d_2 = 3$ the pure fourth region moment is related to the mixed fourth region moment by a factor of 3, for a spherical region, and by 1.8, for a cuboidal region. A spherical region of interest is used in this study although results could be similarly obtained for a cuboidal region of interest.

It is assumed in this study that the region R is not the whole operability region, 0, outside of which physical limits preclude experimentation. R is that region the experimenter is most interested in. The data points in the original undesigned experiment should all be within this region. Using the point in the design furthest from the design center establishes a lower bound on the radius of R. The region of interest may extend beyond this bound but should not be inside of it. For example, in the use of data set B with two variables the point furthest from the center is at a radius of 9.14 while the radius of the region of interest used is 9.55. Care must be taken in actually deciding upon R since different regions will yield different results.

In deciding upon the spherical region of interest, R, the variables ξ_i will be scaled to x_i so that

$$\sum_{i=1}^{k} x_i^2 \leq 1 \tag{3.25}$$

The resulting design moment matrices, M_{ij}, will depend on the value of the radius of R. The region, when scaled, will always have radius 1 so that region moments are unaffected.

Optimization of Bias

For the case where $d_1 = 2$ and $d_2 = 3$; i.e., a second order model used as the approximating polynomial when the true response is third order, B is minimized when design moments through order 5 are equal to the region moments. For N runs of the experiment this can be expressed as

$$[i] = 0 \quad \text{or} \quad \sum_{i=1}^{k} \left(\sum_{u=1}^{N} x_{iu}/N \right) = 0 \tag{3.26}$$

$$[ij] = 0 \quad \text{or} \quad \sum_{i=1}^{k} \sum_{j>i}^{k} \left(\sum_{u=1}^{N} \frac{x_{iu}x_{ju}}{N} \right) = 0 \tag{3.27}$$

$$[ii] = 1/(k+2) \quad \text{or} \quad \sum_{i=1}^{k} \left(\sum_{u=1}^{N} \frac{x_{iu}^2}{N} \right) = 1/(k+2) \tag{3.28}$$

$$[ijl] = 0 \quad \text{or} \quad \sum_{i=1}^{k} \sum_{j\geq i}^{k} \sum_{l\geq j}^{k} \left(\sum_{u=1}^{N} \frac{x_{ij}x_{ju}x_{lu}}{N} \right) = 0 \tag{3.29}$$

$$[iiij] = 0 \quad \text{or} \quad \sum_{i=1}^{k} \sum_{j \neq i}^{k} \left(\sum_{u=1}^{N} \frac{x_{ij}{}^{3} x_{ju}}{N} \right) = 0 \qquad (3.30)$$

$$[iijl] = 0 \quad \text{or} \quad \sum_{i=1}^{k} \sum_{j \neq i}^{k} \sum_{\substack{l \neq i \\ l > j}} \left(\sum_{u=1}^{N} \frac{x_{iu}{}^{2} x_{ju} x_{lu}}{N} \right) = 0 \qquad (3.31)$$

$$[ijlm] = 0 \quad \text{or} \quad \sum_{i=1}^{k} \sum_{j>i}^{k} \sum_{l>j}^{k} \sum_{p>l}^{k} \left(\sum_{u=1}^{N} \frac{x_{iu} x_{ju} x_{lu} x_{pu}}{N} \right) = 0 \qquad (3.32)$$

$$[iijj] = 1/(k + 2)(k + 4) \quad \text{or} \quad \sum_{i=1}^{k} \sum_{j>i}^{k} \left(\sum_{u=1}^{N} \frac{x_{iu}{}^{2} x_{ju}{}^{2}}{N} \right) \qquad (3.33)$$

$$= 1/(k + 2)(k + 4)$$

$$[iiii] = 3/(k + 2)(k + 4) \quad \text{or} \quad \sum_{i=1}^{k} \left(\sum_{u=1}^{N} \frac{x_{iu}{}^{4}}{} \right) = \qquad (3.34)$$

$$= 3/(k + 2)(k + 4)$$

$$[ijlpq] = 0 \quad \text{or} \quad \sum_{i=1}^{k} \sum_{j \geq i}^{k} \sum_{l \geq j}^{k} \sum_{p \geq l}^{k} \sum_{q \geq p} \left(\sum_{u=1}^{N} \frac{x_{iu} x_{ju} x_{lu} x_{pu} x_{qu}}{N} \right) = 0 \qquad (3.35)$$

The use of square bracket notation for design moments and the right-hand side region moment expressions are explained in Appendix A.

These non-linear, simultaneous equations can be formulated into a single objective function. That is, B is minimized when F is zero and

$$F(\underset{\sim}{x}) = \sum_{i=1}^{k} \left(\sum_{u=1}^{N} x_{iu} \right)^{2} + \sum_{i=1}^{k} \sum_{j>i}^{k} \left(\sum_{u=1}^{N} x_{iu} x_{ju} \right)^{2} \quad \text{(Continued)} \qquad (3.36)$$

$$+ \sum_{i=1}^{k}\left(\sum_{u=1}^{N} x_{iu}^{2} - N/(k+2)\right)^{2} + \sum_{i=1}^{k}\sum_{j\geq j}^{k}\sum_{l\geq j}^{k}\left(\sum_{u=1}^{N} x_{iu}x_{ju}x_{lu}\right)^{2}$$

$$+ \sum_{i=1}^{k}\sum_{j\neq i}^{k}\left(\sum_{u=1}^{N} x_{iu}^{3}x_{ju}\right)^{2} + \sum_{i=1}^{k}\sum_{j\neq i}^{k}\sum_{\substack{l\neq i\\ l>j}}^{k}\left(\sum_{u=1}^{N} x_{iu}^{2}x_{ju}x_{lu}\right)^{2}$$

$$+ \sum_{i=1}^{k}\sum_{j\neq i}^{k}\sum_{l>j}^{k}\sum_{p>l}\left(\sum_{u-1}^{N} x_{iu}x_{ju}x_{lu}x_{pu}\right)^{2} + \sum_{i=1}^{k}\sum_{j>i}^{k}$$

$$\left(\sum_{u=1}^{N} x_{iu}^{2}x_{ju}^{2} - N/(k+2)(k+4)\right)^{2} + \sum_{i=1}^{k}$$

$$\left(\sum_{u=1}^{N} x_{iu}^{4} - 3/N(k+2)(k=4)\right)^{2} + \sum_{i=1}^{k}\sum_{j\geq i}^{k}\sum_{l\geq j}^{k}\sum_{p\geq l}^{k}\sum_{q\geq p}^{k}$$

$$\left(\sum_{u=1}^{N} x_{iu}x_{ju}x_{lu}x_{pu}x_{qu}\right)^{2}$$

In N runs of the experiment have been decided upon, then the values of the x_{iu}, $u = 1,\ldots,N$, are fixed. Let

$$c_i = \sum_{u=1}^{N} x_{iu} \quad , \tag{3.37}$$

$$c_{ij} = \sum_{u=1}^{N} x_{iu}x_{ju} \quad , \tag{3.38}$$

$$c_{ijl} = \sum_{u=1}^{N} x_{iu}x_{ju}x_{lu} \quad , \tag{3.39}$$

$$c_{ijlp} = \sum_{u=1}^{N} x_{iu}x_{ju}x_{lu}x_{pu} \quad , \tag{3.40}$$

and

$$c_{ijlpq} = \sum_{u=1}^{N} x_{iu}x_{ju}x_{lu}x_{pu}x_{qu} \quad , \tag{3.41}$$

F can now be written

$$F(\underline{x}) = \sum_{i=1}^{k}\left(c_i + \sum_{u=N+1}^{W} x_{iu}\right)^2 + \sum_{i=1}^{k}\sum_{j>i}^{k}\left(c_{ij} + \sum_{u=N+1}^{W} x_{iu}x_{ju}\right)^2 \tag{3.42}$$

$$+ \sum_{i=1}^{k}\left(c_{ii} - N/(k+2) + \sum_{u=N+1}^{W} x_{iu}^{2} - m/(k+2)\right)^2$$

$$+ \sum_{i=1}^{k}\sum_{j\geq i}^{k}\sum_{l\geq j}^{k}\left(c_{ijl} + \sum_{u=N+1}^{W} x_{iu}x_{ju}x_{lu}\right)^2 + \sum_{i=1}^{k}\sum_{j\neq l}^{k}$$

$$\left(c_{iiij} + \sum_{u=N+1}^{W} x_{iu}^{3}x_{ju}\right)^2 + \sum_{i=1}^{k}\sum_{j\neq i}^{k}\sum_{\substack{l\neq i\\ l>j}}\left(c_{iijl} + \sum_{u=N+1}^{W} x_{iu}x_{ju}x_{lu}\right)^2$$

$$+ \sum_{i=1}^{k}\sum_{j>i}^{k}\sum_{l>j}\sum_{p>l}\left(c_{ijlp} + \sum_{u=N+1}^{W} x_{iu}x_{ju}x_{lu}x_{pu}\right)^2 + \sum_{i=1}^{k}\sum_{j>i}^{k}$$

$$\left(c_{iijj} - N/(k+2)(k+4) + \sum_{u=N+1}^{W} x_{iu}^{2}x_{ju}^{2} - m/(k+2)(k+4)\right)^2$$

$$+ \sum_{i=1}^{k}\left(c_{iiii} - N/(k+2)(k+4) + \sum_{u=N+1}^{W} x_{iu}^{4} - m/(k+2)(k+4)\right)^2$$

$$+ \sum_{i=1}^{k} \sum_{j\geq i}^{k} \sum_{l\geq j}^{k} \sum_{p\geq l}^{k} \sum_{q\geq p}^{k} \left(c_{ijlpq} + \sum_{u=N+1}^{W} x_{iu} x_{ju} x_{lu} x_{pu} x_{qu} \right)^2 ,$$

where $W = N + m$, and m is the number of additional runs selected at one time.

Equations 3.26 to 3.29 are the moment conditions for the case of $d_1 = 1$, $d_2 = 2$; i.e., a first order model used when the true response is second order. Therefore, in the first order case, F can be written as

$$F(\underline{x}) = \sum_{i=1}^{k} \left(c_i + \sum_{u=N+1}^{W} x_{iu} \right)^2 + \sum_{i=1}^{k} \sum_{j>i}^{k} \left(c_{ij} + \sum_{u=N+1}^{W} x_{iu} x_{ju} \right)^2 \tag{3.43}$$

$$+ \sum_{i=1}^{k} \left(c_{ii} - N/(k + 2) + \sum_{u=N+1}^{W} x_{iu}^2 - m/(k + 2) \right)^2$$

$$+ \sum_{i=1}^{k} \sum_{j\geq i}^{k} \sum_{l\geq j}^{k} \left(c_{ijl} + \sum_{u=N+1}^{W} x_{iu} x_{ju} x_{lu} \right)^2$$

F is minimized by selecting the x_{iu}, $u = N + 1,\ldots,W$. These values then become the additional experimental runs which minimize bias. The procedure is repeated until the maximum number of additional runs allowed have been decided upon.

In order to minimize F, the Hooke and Jeeves pattern search algorithm was selected. McCaa's (15) work indicated that F might not be well-behaved, hence pattern search, which is known for its good performance characteristics on irregular functions, was selected. This is a derivative-free method using a search technique which evaluates the functional value in a predetermined pattern to determine a minimum. It is not a global search technique, so the possibility exists that local minima may be found. A complete description of pattern search is given by Hooke and Jeeves (11).

A method for determining additional runs to reduce average squared bias has been described. It remains to develop some measurement of effectiveness. The technique assumes that average bias, B, is of more immediate concern than average variance, V. Values of B and J can only be determined by assuming values for $\underline{\alpha}_2$. For the purpose of this investigation, the $\underline{\beta}_2$ terms are assumed to be equally weighted at a value of 1. $\underline{\beta}_2/\sigma$ ratios are selected to allow the calculation of specific numerical values. The values selected are .5, 1.0, 2.5, 5.0, and 10.0. The lower values may allow some feeling for what happens when V is, in fact, greater than B, while the larger values should ensure B dominates V.

From equation 3.19, it is seen that B is made up of two components. One component, say B_m, involves only region moments defined by equation 3.24 and the other component, say B_c, includes design moments as well as region moments. For a given number of runs, N, the first part becomes the minimum average bias possible and is not controllable by the experimenter. The worth of any method must then be judged on the basis of its performance with respect to

$$B_c = \underline{\alpha}_2'[(M_{11}^{-1}M_{12} - \mu_{11}^{-1}\mu_{12})\mu_{11}(M_{11}^{-1}M_{12} - \mu_{11}^{-1}\mu_{12})]\underline{\alpha}_2 \qquad (3.44)$$

Since our goal is to reduce average bias, a simple measure of success is the percent reduction in this term. Let R_1 equal percent reduction or

$$R_1 = \frac{B_{co}-B_{ca}}{B_{co}} \times 100\% \tag{3.45}$$

where B_{co} is the value of B_c for the original design, and B_{ca} is the value after augmentation. The same measure can be used for total bias. Let

$$R_2 = \frac{B_o-B_a}{B_o} \times 100\% \tag{3.46}$$

Since B_m, the uncontrollable portion, increases as points are added to the design, a point will be reached where the quantity R_2 begins to drop. This point is definitely reached when B_c is zero but may occur before this if the improvement in B_c is not as great as the increase in B_m due to adding more runs.

R_1 and R_2 do not depend on the choice of $\underline{\beta}_2/\sigma$. This can be seen by letting Q_o represent the terms inside the square brackets in equation 3.44 so that

$$B_o = \underline{\alpha}_{20}' \; Q_o \underline{\alpha}_{20} \quad , \tag{3.47}$$

then

$$R_1 = \frac{\underline{\alpha}_{20}' \; Q_o \underline{\alpha}_{20} - \underline{\alpha}_{2a}' \; Q_a \underline{\alpha}_{2a}}{\underline{\alpha}_{20}' \; Q_o \underline{\alpha}_{20}} \quad . \tag{3.48}$$

Since

$$\underline{\alpha}_{20} = \sqrt{No}\ \underline{\beta}_2/\sigma \quad , \qquad \underline{\alpha}_{2a} = \sqrt{Na}\ \underline{\alpha}_2/\sigma \tag{3.49}$$

with No = N and Na = N + m, then

$$R_1 = \frac{NoQo-NaQa}{N_o Q_o} \tag{3.50}$$

The same result can be shown for R_2.

The value of V is calculated for each design, so that the ratio of variance to bias can be calculated. If average variance is very much greater than average bias, then the technique developed here may not be appropriate. This ratio, of course, depends on the value chosen for $\underline{\beta}_2/\sigma$, but, in practice, the use of a range of values for $\underline{\beta}_2/\sigma$ should give some indication of any potential difficulties.

The value of J is also computed to enable the investigator to examine the effects of the procedure on the mean square error. However, since the value of J depends on $\underline{\beta}_2/\sigma$, it is not used directly as a criterion, and its value can not be used directly to measure results.

In the literature the ratio of the determinant of $(X'X)$ for the augmented design to the determinant of the original $(X'X)$ is used as a measure of improvement for variance. This same ratio is used here to illustrate the effects of the bias-oriented procedure on variance. Although the procedure is concerned with bias, variance also will improve, as the addition of any point causes $|(X'X)|$ to increase. This reduction of variance will, in general, not be optimal.

CHAPTER IV

DISCUSSION OF RESULTS

The objective of this investigation was to develop a technique for repairing undesigned experiments by the selection of additional experimental runs. Improvement in the design was to be measured by the percentage reduction in bias terms. The FORTRAN program in Appendix B was used to select a total of M additional runs for various undesigned experiments, where M is the maximum number of runs allowed to repair the design. The value of M used in this investigation is higher than can be expected in actual application of this technique. This allowed a more complete investigation of the effects of augmentation. If the number of additional runs allowed is not limited to a number that is less than an appropriate designed experiment would require, use of the designed experiment should be considered. Two methods of selecting the additional runs were used. First the additional runs were selected one at a time; i.e., $m = 1$ in equation 3.42. Then using the same data, they were selected in pairs; i.e., $m = 2$. The results are tabulated in Appendix D.

The undesigned experiments came from four data sets. These data sets are referred to as A, B, C, and D for convenient reference and are shown in Tables 2 to 5 in Appendix C.

The pattern search used for minimizing $F(\underline{x})$ was programmed to search the region of interest, R, as well as points outside the region. Use of different starting points resulted in some differences in the additional runs selected, but only minor differences occurred in the value of $F(\underline{x})$. The effect on the reduction of bias was negligible. The results in Appendix D all used starting points of .001 for each transformed variable. The use of the exact center of the scaled region of interest caused the program to enter a continuous loop. Any point within an ε distance of the center should work well. The center of the region is used since there is no theoretical basis for selecting starting points for an undesigned experiment.

Table 1 summarizes the results in Appendix C. The optimum R_2 is the maximum reduction of average bias achieved for the design considered. The value was normally reached before M runs were added. This suggests that an interactive computer program could be used via a terminal. The operator would stop selecting further runs when R_2 decreased or M runs had been selected. If all M runs are added, R_2 is the percentage reduction of bias but not optimal.

There appears to be no advantage in terms of bias in selecting runs singly or in pairs. In most cases one additional run is saved by using single selection rather than paired selection, but data set C shows this is not always the case. The difference in R_2 between the two methods is small, and the cost of making additional runs would have to be considered before a clear-cut choice could be made. Use of both techniques as shown here would depend on the cost of computer time involved. Selecting all M runs at one time is not a good procedure, as the optimal R_2 could be masked.

Table 1 also indicates two cases where there was no improvement in average bias. Using data set B, and considering a first order model

Table 1. Summary of Results

Data Used	Variables	Order of Model Used	Maximum Additional Runs Allowed	m	Optimum R_2	No. of Runs Added at Optimum R_2
A	2	First	4	1	52.2	1
				2	49.7	2
B	2	First	10	1	85.7	2
				2	85.8	2
B	3	First	10	1	-15.6	10
				2	- 5.0	8
B	2	Second	10	1	88.1	5
				2	87.6	6
B	3	Second	10	1	87.8	3
				2	82.1	4
C	3	Second	10	1	20.2	3
				2	20.0	2
D_1	2	Second	6	1	60.4	5
				2	60.1	6
D_2	2	Second	6	1	- .3	1
				2	-12.0	2

and three variables, Table 14 in Appendix C shows that controllable bias increases with the first additional run even though $F(\underline{x})$ decreases. Eventually controllable bias is decreased, but by this time uncontrollable bias has increased more than the maximum possible decrease in controllable bias. The decrease in $F(\underline{x})$ indicates that the design moments have changed but not enough to compensate for the change in $\underline{\alpha}_2$, which increases with N, as shown in equation 3.51. The data is spread out enough in the region of interest to protect against the true model being second order. Another approach should be used for repairing this undesigned experiment.

Two regions of interest were used with data set D. For D_1, the region of interest is a circle of radius 1.41, as used by Hebble and Mitchell (9). Reductions in average bias of 60% are achieved, and a comparison of the additional runs selected with those considering variance is found in Figure 1. Using this same data with a region of interest that includes all the points in D_1, results in a circle of radius 1.7. In this case controllable bias is reduced but not enough to overcome the increase in uncontrollable bias, as shown in Table 34, Appendix C. This imperfect central composite design with two center points is not a good candidate for bias consideration and should be examined with respect to variance.

Figure 2 presents a comparison between Dykstra's choice of additional runs based on variance considerations and those selected here based on bias. A second order model is used with data set B. The points selected are quite different. Considering bias with either m = 1 or m = 2, additional runs are taken in the second quadrant. Apparently

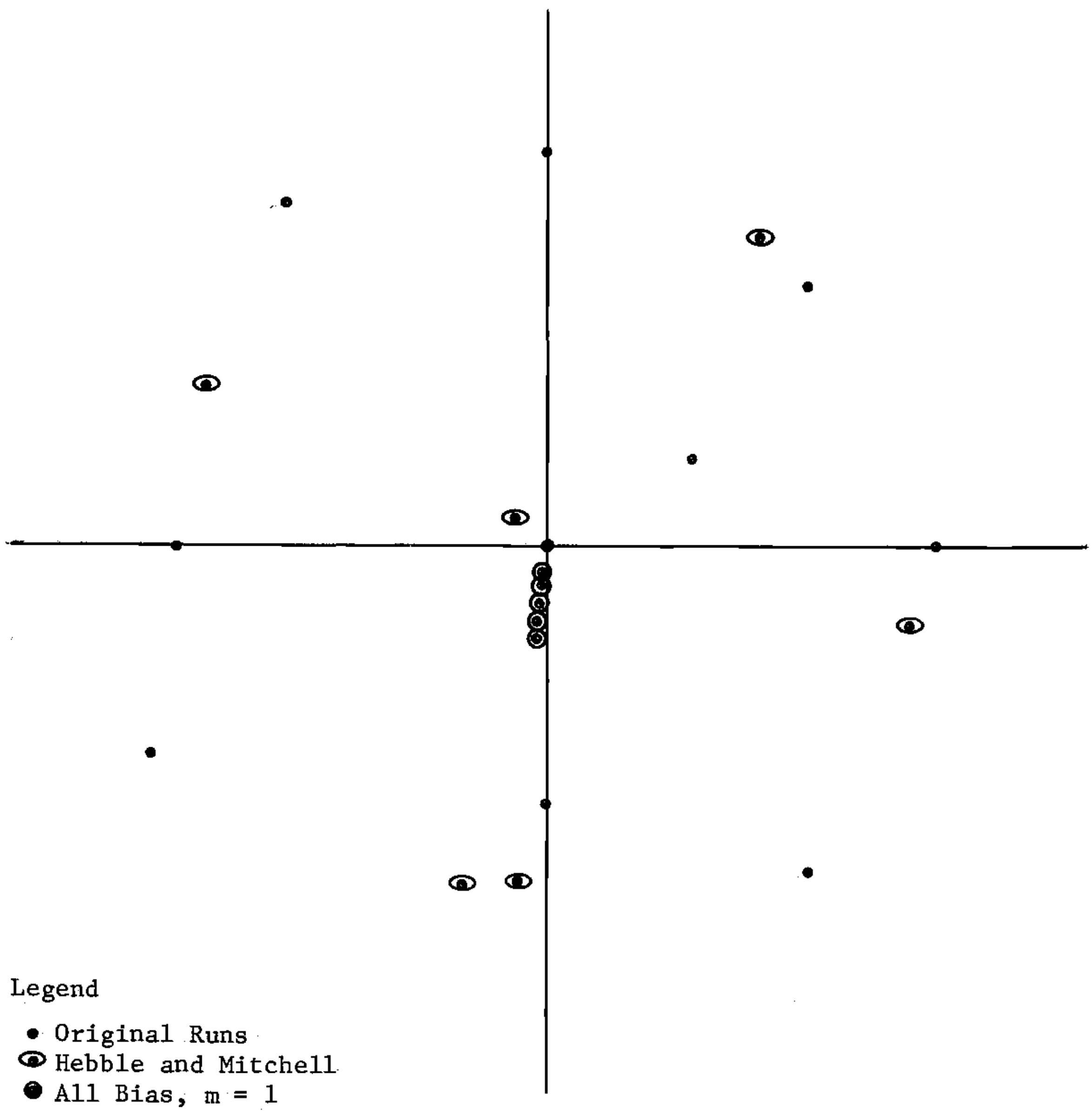

Figure 1. Additional Runs, Data Set D, R = 1.41

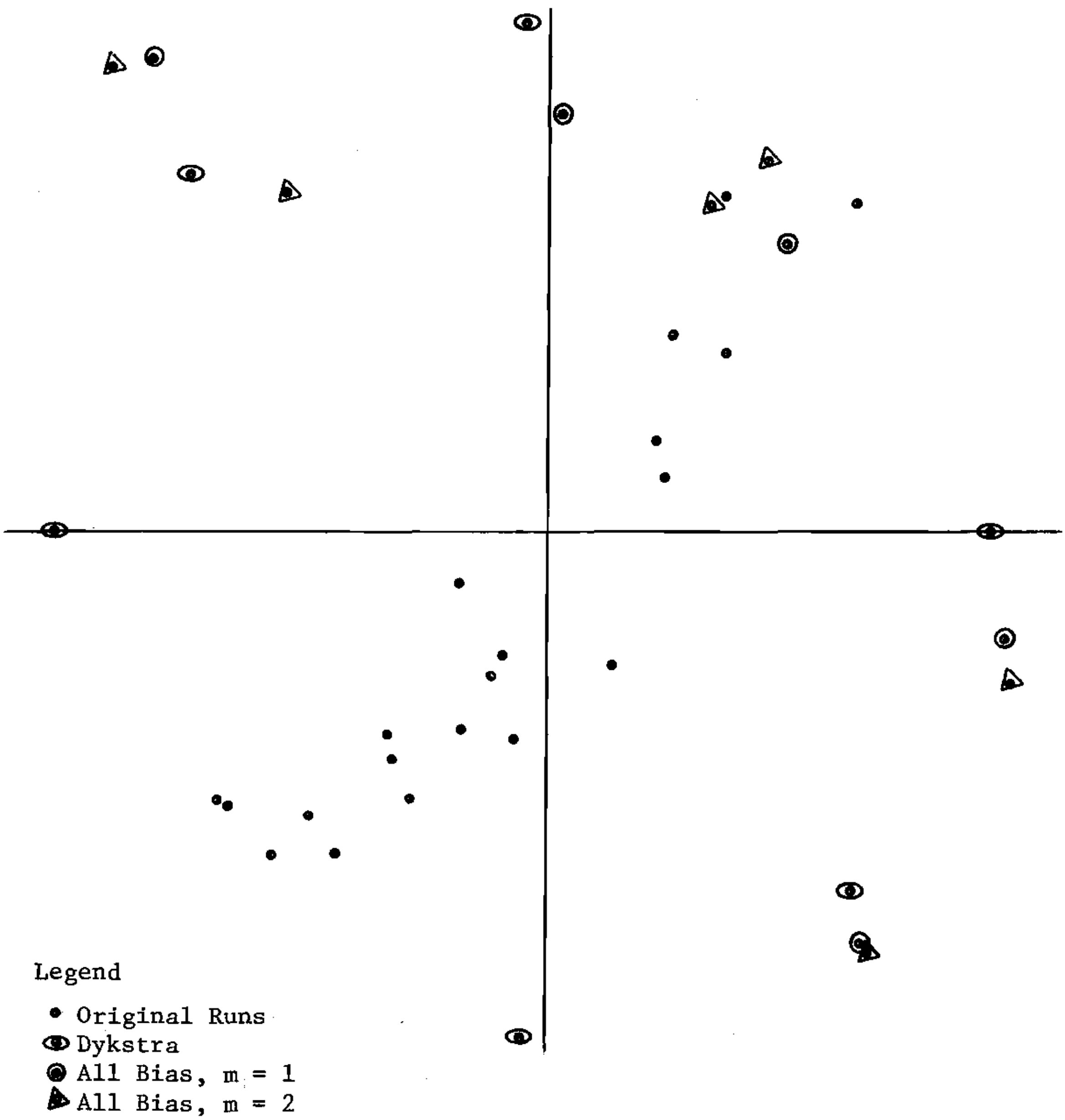

Figure 2. Additional Runs, Data Set B, Second Order

after balancing the experiment in the second and fourth quadrant, the heavy concentration of points in the third quadrant becomes important and is offset by adding runs in the first quadrant. Comparing the runs added for m = 1 and m = 2, those added in pairs balance each other in one variable or another within the pair. Those selected singly do not follow this pattern.

Use of this data allows comparison of the effect of adding runs by the method in this investigation as opposed to variance through comparison of the ratio's of $|(X'X)|$. Dykstra, after adding six runs, has a ratio of 3.39×10^5. In this study adding six points, m = 2, results in a ratio of 1.98×10^5. For five additional runs Dykstra has an $|(X'X)|$ ratio of 1.75×10^5, while our method with m = 1, has a ratio of 1.2×10^5. Thus, considering bias alone, this procedure has also had an affect on variance that is close to that obtained by considering variance alone.

The value of average mean square error was computed for each design in order to examine the effects of this procedure on its value. For those cases where the procedure is recommended for use J is found to decrease initially. Depending on the value of $\underline{\beta}_2/\sigma$ used, which determine whether variance or bias is the largest contributor to the value of J, it may reach a minimum. This happens when the value of bias is increasing due to an additional run more than variance is decreased. For $\underline{\beta}_2/\sigma = 10.0$ it occurs for every experiment where the procedure is recommended. Other values of $\underline{\beta}_2/\sigma$ do not always exhibit this effect. The number of additional runs necessary to reach minimum J also depends on $\underline{\beta}_2/\sigma$, but in no case does J minimum occur in less additional runs than required for optimum R_2. This is to be expected since bias does not increase in value until optimum R_2 is reached. The calculation of variance to bias ratios proved meaningless for determining when the technique would not work. The technique worked well even when V/B appeared high, as in Table 22, yet failed when V/B appeared average, as in Table 14, at the optimal number of additional runs for bias.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

Three significant conclusions are indicated by the results of this study. They are:

1. Undesigned response surface experiments may be improved with respect to average squared bias by the sequential addition of runs that are selected to minimize the function $F(\underline{x})$.

2. The method used indicates when improvement in average bias is not possible.

3. The simultaneous selection of all available additional runs rather than sequential selection will not, in general, result in the optimum reduction of average bias.

Recommendations

During this investigation several areas were encountered that merit further study. A number of undesigned experiments should be examined to determine if the optimal reductions of average bias using $m = 1$ and $m = 2$ are always achieved within one additional run of each other. If so, a technique determining an optimal reduction in average bias with $m = 2$ could be used first. Then, using $m = 1$, a search could be made to determine if increasing or decreasing the number of runs would result in a greater reduction in bias.

The potential problems due to taking the additional runs at a different time and under different experimental conditions have not been considered in this investigation. The use of standard techniques for blocking should be examined.

The matrix $(\mu_{11}M_{11}^{-1})$ is used in determining the average variance. This matrix is not symetric in an undesigned experiment and is not similar to the usual variance-covariance matrix. The meaning of the off-diagonal terms in this matrix is not clear and should be studied further.

This investigation showed that an optimal reduction in average bias is often reached before the total number of additional runs available is used. Using a variance criterion to select any remaining runs could offer further improvement in the design. The effect of these additional runs on bias may be detrimental, however, and a measure of design improvement may be difficult to determine.

APPENDICES

APPENDIX A

DESIGN MOMENTS AND REGION MOMENTS

The design moments matrix is defined to be

$$N^{-1}(X_1X_1) \tag{A.1}$$

where N is the total number of experimental runs in the design. In the first order case

$$N^{-1}X_1X_1 = (\underline{1},\underline{x}_1,\underline{x}_2,\ldots\underline{x}_k)\begin{pmatrix}\underline{1}\\ \underline{x}_1\\ \underline{x}_2\\ \vdots\\ \underline{x}_k\end{pmatrix} \tag{A.2}$$

$$= \begin{bmatrix} 1 & [1] & [2] & \ldots & [k] \\ & [11] & [12] & \ldots & [1k] \\ & & [22] & \ldots & [2k] \\ & & & & \vdots \\ \text{Sym.} & & & & [kk] \end{bmatrix}$$

where the square bracket notation refers to first order moments

$$[i] = \frac{1}{n}\sum_{u=1}^{n} x_{iu} \tag{A.3}$$

and second order moments

$$[ii] = \frac{1}{n}\sum_{u=1}^{n} x_{iu}x_{ju} \tag{A.4}$$

For the second order case for k = 2

$$N^{-1}X_1X_1 = \begin{bmatrix} 1 & [1] & [2] & [11] & [22] & [12] \\ & [11] & [12] & [111] & [122] & [112] \\ & & [22] & [211] & [222] & [122] \\ & & & [1111] & [1122] & [1212] \\ & & & & [2222] & [1222] \\ \text{Sym.} & & & & & [1122] \end{bmatrix} \tag{A.5}$$

An example of a third order moment being

$$[112] = \frac{1}{n} \sum_{u=1}^{n} x_{1u}^2 x_{2u} \tag{A.6}$$

and an example of fourth order moments being

$$[1122] = \frac{1}{n} \sum_{u=1}^{n} x_{1u}^2 x_{2u} \quad . \tag{A.7}$$

Similarly fifth order moments would be

$$[ijklm] = \frac{1}{n} \sum_{u=1}^{n} x_{iu} x_{ju} x_{ku} x_{lu} x_{mu} \quad . \tag{A.8}$$

Region moments are of the form

$$\mu_{12} = \int_R \underline{x}_1 \underline{x}_2 d\underline{x} \tag{A.9}$$

where $\underline{x}_1$ is from the postulated model

$$y = \underline{x}_1 \underline{\beta} \tag{A.10}$$

and $\underline{x}_2$ is from the assumed true model

$$y = \underline{x}_1\beta_1 + \underline{x}_2\beta_2 \quad . \tag{A.11}$$

These may be evaluated using

$$\int_R x_1^{\delta_1} x_2^{\delta_2} \ldots x_k^{\delta_k} \, dx = \tag{A.12}$$

$$\frac{\Gamma\left(\frac{\delta_1+1}{2}\right)\Gamma\left(\frac{\delta_2+1}{2}\right) \cdot \cdot \cdot \Gamma\left(\frac{\delta_k+1}{2}\right)}{\Gamma\left\{\frac{\sum_{i=1}^{k} (\delta_k+1)}{2} + 1\right\}}$$

where Γ is the usual gamma function, that is,

$$(P) = \int_0^\infty v^{p-1}e^{-v}dv \quad . \tag{A.13}$$

APPENDIX B

FORTRAN PROGRAM

```
C     REPAIRING UNDESIGNED EXPERIMENTS
      REAL MU11,MU
      DIMENSION MU11(21,21),MU(21,35),CON(35,35)
      DIMENSION SM1(21,21),SM12(21,35),BB(5),CB(5)
      DIMENSION D(5,36),XS(5,36),S(5),AVG(5),DSX(5,36),DEL(5)
      DIMENSION XSX(36,21),XSXT(21,36),XSTXS(21,21)
      DIMENSION X2(36,35),DETXS(17)
      DIMENSION V(2),JC(2)
      DIMENSION BBA(17,5),CBA(17,5)
      DIMENSION BIAS(17,5),PERRED(17,5),BIARED(17,5),JAUG(17)
      DIMENSION VVAR(17)
      DIMENSION  VAL(17),NFE(17),RATIO(17)
      DIMENSION VAR(17),VA(21,21),VBR(17,5),AJBV(17,5)
      COMMON/ A1/K,NM,KEY,KORDER
      COMMON/A2/EMAX(5),EMIN(5),BASEN(5,36),NN,DELTA(5),AP
      COMMON/A3/FN,NE,M,MM
      COMMON/A4/ R(5,2)
      COMMON/A5/K0,K1,K2,K3,K4,K5,K11,K14,II,KS
      COMMON/A6/BETSIG
C     N IS NO. OF POINTS IN D
C     M IS NO. OF POINTS SELECTED AT A TIME
C     KEY IS USED FOR SELECTED OUTPUT
C     KORDER IS 4 FOR FIRST ORDER, 10 FOR SECOND
C     AP IS TOTAL NO. OF POINTS TO BE ADDED
C     K IS THE NO. OF VARIABLES
C     D IS THE DESIGN MATRIX
C     DEL IS THE ACCURACY DESIRED ON THE VARIABLES
C     R IS THE REGION OF INTEREST OF EACH VARIABLE
C     BASEN IS THE START POINT FOR EACH VARIABLE
C      SCALEA SCALES THE DESIGN TO THE REGION +1,-1
C     FUNVAL EVALUATES THE FUNCTION F(X)
C     XMTX DETERMINES THE X1 AND X2 MATRICES
C     COMU COMPUTES THE REGION MOMENTS
C     MINMIZ IS THE HOOKE AND JEEVES ROUTINE
C     MXTRN,MXMLT,GJR ARE SUBROUTINES FOR MATRIX
C         TRANSPOSITION, MULTIPLICATION,AND INVERSE/DETERMINANT
C     BBB CALCULATES BIAS VALUES
      INTEGER U,AP
      READ(5,999) IRUNS
      DO 4 I=1,5
      DO 4 U=1,36
    4 D(I,U)=0.
      READ(5,1000) N,M,KEY,KORDER,AP,K,BETSIG
      DO 1 J=1,N
    1 READ(5,1001)(D(I,J),I=1,K)
      READ(5,1001) (DEL(I),I=1,K)
      DO 100 IR=1,IRUNS
      DO 2 I=1,21
      VVAR(I)=0.
    2 VAR(I)=0.
      READ(5,1000) N,M,KEY,KORDER,AP,K,BETSIG
      READ(5,2050) ((R(I,J),J=1,2),I=1,K)
      READ(5,1001) (EMAX(I),I=1,K)
      DO 5 I=1,K
      EMIN(I)=-EMAX(I)
    5 DELTA(I)=1.6
      NN=N+1
      MM=N+AP
      JI=1
      IN=N
    3 NM=N
      BNM=NM
      IAP=AP
      KN=0
      DO 7 I=1,K
      DO 7 J=1,MM
    7 BASEN(I,J)=0.001
   10 CALL SCALEA(D,XS,S,AVG)
      CALL FUNVAL(XS,SOLVAL)
      VAL(1)=SOLVAL
```

```
      NFE(1)=0
      CALL XMTX(XS,XSX,X2)
      CALL COMU(MU11,MU,CON)
   15 FVALUE=SCLVAL
      NAP=0
      AP=M
      IF(JI.EQ.1) GO TO 760
   41 CALL MINMIZ(XS,DEL,FVALUE)
      NN=NN+M
      VAL(JI)=FVALUE
      NFE(JI)=NE
      CALL XMTX(XS,XSX,X2)
  760 CALL MXTRN(XSX,XSXT,NM,K4,36,21)
      CALL MXMLT(XSXT,XSX,XSTXS,K4,NM,K4,21,36)
      V(1)=3.
      CALL GJR(XSTXS,21,21,K4,K4,$60,JC,V)
      DETXS(JI)=V(1)*EXP(V(2))
      RATIO(JI)=DETXS(JI)/DETXS(1)
      CALL MXMLT(XSXT,X2,SM12,K4,NM,K14,21,36)
      DO 800 I=1,K4
      DO 800 J=1,K14
  800 SM12(I,J)=SM12(I,J)/NM
      DO 202 I=1,K4
      DO 202 J=1,K4
  202 SM1(I,J)=BNM*XSTXS(I,J)
      CALL BBB(MU11,MU,CON,SM12,SM1,BB,CB)
      CALL MXMLT(SM1,MU11,VA,K4,K4,K4,21,21)
      DO 680 I=1,K4
      DO 681 J=1,K4
  681 VVAR(JI)=VVAR(JI)+VA(I,J)
  680 VAR(JI)=VAR(JI)+VA(I,I)
      DO 690 I=1,5
      BBA(JI,I)=BB(I)
      CBA(JI,I)=CB(I)
      BIAS(JI,I)=BBA(JI,I)+CBA(JI,I)
      PERRED(JI,I)=((BBA(1,I)-BBA(JI,I))/BBA(1,I))*100
      BIARED(JI,I)=((BIAS(1,I)-BIAS(JI,I))/BIAS(1,I))*100
      JAUG(JI)=JI-1+IN
      VBR(JI,I)=VAR(JI)/BIAS(JI,I)
      AJBV(JI,I)=BIAS(JI,I)+VAR(JI)
  690 CONTINUE
      NAP=NAP+M
      IF(NAP.GT.IAP) GO TO 55
      NM=NM+M
      JI=JI+M
      GO TO 41
   55 CALL DESCL(XS,S,AVG,DSX)
      IF(KORDER.EQ.4) GO TO 21
      WRITE(6,4002) N,K,M
      WRITE(6,1010)
      GO TO 11
   21 WRITE(6,4001) N,K,M
      WRITE(6,1010)
   11 WRITE(6,4000) BMAX(1),B(1,1),DELTA(1)
      KAP=IAP+1
      WRITE( 6,4004)
      WRITE(6,1010)
      WRITE(6,4003)
      WRITE(6,1010)
      DO 12 I=1,KAP,M
   12 WRITE(6,4005) JAUG(I),(CBA(I,J),J=1,5)
      WRITE(6,1010)
   26 WRITE(6,4006)
      WRITE(6,1010)
      WRITE(6,4030)
      WRITE(6,1010)
      DO 13 I=1,KAP,M
   13 WRITE(6,4007) JAUG(I),(BBA(I,J),J=1,5),PERRED(I,1)
      WRITE(6,1010)
      WRITE(6,4008)
```

```
      WRITE(6,1010)
      WRITE(6 ,4031)
      WRITE(6,1010)
      DO 14 I=1,KAP,M
   14 WRITE(6,4007) JAUG(I),(BIAS(I,J),J=1,5),BIARED(I,1)
      WRITE(6,1010)
      WRITE(6,905)
   25 WRITE(6,4017)
      WRITE(6,1010)
      WRITE(6,4032)
      WRITE(6,1010)
      DO 23 I=1,KAP,M
   23 WRITE(6,4007) JAUG(I),(VBR(I,J),J=1,5),VAR(I)
      WRITE(6,1010)
      WRITE(6,4018)
      WRITE(6,1010)
      WRITE(6,4003)
      WRITE(6,1010)
      DO 24 I=1,KAP,M
   24 WRITE(6,4005) JAUG(I),(AJBV(I,J),J=1,5)
      WRITE(6 ,1010)
      WRITE(6,905)
      WRITE(6,906)
      WRITE(6,906)
      WRITE(6,1010)
      IF(K.EQ.3) GO TO 19
      WRITE(6,4010)
      WRITE(6,1010)
   17 WRITE(6,4012) JAUG(1),VAL(1),RATIO(1)
      IF(M.EQ.2) GO TO 550
      IF(K.EQ.3) GO TO 18
      DO 22 I=2,KAP
      IU=I+IN-1
   22 WRITE(6,4014) JAUG(I),VAL(I),RATIO(I),PERRED(I,2),
     1BIARED(I,2),(DSX(J,IU),J=1,2)
      WRITE(6,1010)
      GO TO 100
   18 DO 16 I=2,KAP
      IU=I+IN-1
   16 WRITE( 6,4013) JAUG(I),VAL(I),RATIO(I),PERRED(I,2),
     1BIARED(I,2),(DSX(J,IU),J=1,3)
      WRITE(6,1010)
      GO TO 100
  550 IF(K.EQ.3) GO TO 580
      DO 551 I=3,KAP,M
      IU=I+IN-2
      WRITE(6,4019) (DSX(J,IU),J=1,2)
      IU=IU+1
  551 WRITE(6,4014) JAUG(I),VAL(I),RATIO(I),PERRED(I,2),
     1BIARED(I,2),(DSX(J,IU),J=1,2)
      WRITE(6,1010)
      GO TO 100
  580 DO 555 I=3,KAP,M
      IU=I+IN-2
      WRITE( 6,4020) (DSX(J,IU),J=1,3)
      IU=IU+1
  555 WRITE(6,4013) JAUG(I),VAL(I),RATIO(I),PERRED(I,2),
     1BIARED(I,2),(DSX(J,IU),J=1,3)
      WRITE(6,1010)
      GO TO 100
   19 WRITE(6,4011)
      WRITE(6,1010)
      GO TO  17
   60 WRITE(6,3020) JC(1),V(1),V(2)
      WRITE(6,905)
      DO 501 I=1,N
  501 WRITE(6,1008)(XSX(I,J),J=1,K4)
      WRITE(6,906)
      DO 506 I=1,N
  506 WRITE(6,1008)(X2(I,J),J=1,K14)
```

```
      WRITE(6,906)
      DO 507 I=1,K4
  507 WRITE(6,1008)(VA(I,J),J=1,K4)
      WRITE(6,906)
      DO 505 I=1,K4
  505 WRITE(6,1008)(SM12(I,J),J=1,K14)
      DO 504 I=1,K4
  504 WRITE(6,1008)(XSTXS(I,J),J=1,K4)
      WRITE(6,905)
  502 WRITE(6,1008)(XSX(NM,J),J=1,K4)
      WRITE(6,906)
  503 WRITE(6,1008)(X2(NM,J),J=1,K14)
  100 CONTINUE
      WRITE(6 ,905)
  900 FORMAT(F15.2)
  905 FORMAT (1H1)
  906 FORMAT(1H0)
  999 FORMAT(I10)
 1000 FORMAT(6I5,F5.2)
 1001 FORMAT(5F15.5)
 1004 FORMAT(5F10.5)
 1008 FORMAT(10F10.2)
 1010 FORMAT(1H ,T20,40H----------------------------------------,
     230H------------------------------)
 2080 FORMAT(10F7.2)
 3020 FORMAT(/T10,14HAT GJR JC(1)= ,F5.2,4HV = ,F5.2,F10.3)
 4000 FORMAT(3F10.2)
 4001 FORMAT(1H1,3X,I2,14H ORIGINAL RUNS,I2,10H VARIABLES,
     113H  FIRST ORDER,4H  M=,I2)
 4002 FORMAT(1H1,3X,I2,14H ORIGINAL RUNS,I2,10H VARIABLES,
     214H  SECOND ORDER,4H  M=,I2)
 4003 FORMAT(T21,13H RUN B/S=0.5 ,T41,3H1.0,T50,3H2.5,
     3T59,4H5.0 ,T69,4H10.0)
 4004 FORMAT(1H1,T35,23HVALUE OF UNCONTROLLABLE,
     416H PORTION OF BIAS)
 4005 FORMAT(T22,I2,T25,F8.1,T36,F8.1,T45,F8.1,T55,F8.1,T65,F8.1)
 4006 FORMAT(1H0,T26,27HVALUE AND PERCENT REDUCTION,
     532H OF CONTROLLABLE PORTION OF BIAS)
 4007 FORMAT(T22,I2,T25,F8.1,T36,F8.1,T45,F8.1,T55,F8.1,T65,F8.1,
     7T75,F8.1)
 4008 FORMAT(1H0,T37,35HVALUE AND PERCENT REDUCTION OF BIAS)
 4010 FORMAT(T21,12HRUN FUNCTION,T38,3HX'X,T49,2HR1,
     5T58,2HR2,T69,2HX1,T81,2HX2)
 4011 FORMAT(T21,12HRUN FUNCTION,T38,3HX'X,T49,2HR1,
     5T58,2HR2,T66,2HX1,T75,2HX2,T83,2HX3)
 4012 FORMAT(T21,I2,T23,F9.1,2X,1PE9.3)
 4013 FORMAT(T21,I2,T23,F9.1,2X,1PE9.3,1X,0PF8.1,
     31X,F8.1,F9.3,F9.3,F9.4)
 4014 FORMAT(T21,I2,T23,F9.1,2X,1PE9.3,1X,0PF8.1,
     31X,F8.1,T64,F9.3,T76,F9.3)
 4017 FORMAT(1H0,T34,16HVARIANCE TO BIAS,
     925H RATIO, VALUE OF VARIANCE)
 4018 FORMAT(1H0,T50,10HVALUE OF J)
 4019 FORMAT(T64,F9.3,T76,F9.3)
 4020 FORMAT(T61,F9.3,T70,F9.3,T79,F9.4)
 4030 FORMAT(T21,13H RUN B/S=0.5 ,T41,3H1.0,T50,3H2.5,
     3T59,4H5.0 ,T69,4H10.0,T80,2HR1)
 4031 FORMAT(T21,13H RUN B/S=0.5 ,T41,3H1.0,T50,3H2.5,
     3T59,4H5.0 ,T69,4H10.0,T80,2HR2)
 4032 FORMAT(T21,13H RUN B/S=0.5 ,T41,3H1.0,T50,3H2.5,
     3T59,4H5.0 ,T69,4H10.0,T77,8HVARIANCE)
      END
```

```
      SUBROUTINE BBB(MU11,MU,CON,SM12,SM1,BB,CB)
      REAL MU11,MU
      DIMENS ION MU11(21,21),MU(21,35),CON(35,35)
      DIMENSION SM1(21,21),SM12(21,35),BB(5),CB(5)
      DIMENSION  B(1,1),C(1,1)
      DIMENS ION ALIAS(21,35),DMR(21,35), DMRT(35,21)
      DIMENS ION CCON(1,35),ALPHA2(1,35)
      DIME NSION DMU(35,21),Q(35,35),ALPHAT(35,1),AQ(1,35)
      COMMON /A1/K,NM,KEY,KORDER
      COMMON/A5/K0,K1,K2,K3,K4,K5,K11,K14,I1,KS
      COMMON/ A6/BETSIG
      ANM=NM
      BETS IG=0.5
      KB=1
      IJ=1
  110 DO 11 J=1,K14
   11 ALPHA2(1,J)=(ANM**.5)*BETSIG
      CALL MXMLT(SM1,SM12,ALIAS,K4,K4,K14,21,21)
      CALL MXSUB(ALIAS,MU,DMR,K4,K14,21)
      CALL MXTRN(ALPHA2,ALPHAT,1,K14,1,35)
      CALL MXTRN(DMR,DMRT,K4,K14,21,35)
   50 CALL MXMLT(DMRT,MU11,DMU,K14,K4,K4,35,21)
      CALL MXMLT(DMU,DMR,Q,K14,K4,K14,35,21)
      CALL MXMLT(ALPHA2,Q,AQ,1,K14,K14,1,35)
      CALL MXMLT(AQ,ALPHAT,B,1,K14,1,1,35)
  180 CALL MXMLT(ALPHA2,CON,CCON,1,K14,K14,1,35)
      CALL MXMLT(CCON,ALPHAT,C,1,K14,1,1,35)
      BB(KB)=B(1,1)
      CB(KB)=C(1,1)
      KB=KB+1
      IF(BETSIG - 1.) 115,120,300
  115 BETSIG=1.
      GO TO 110
  120 BETSIG =2.5
      IF(KEY.EQ.1) GO TO 110
      WRITE(6,1001)
      WRITE(6 ,1000) (ALPHA2(1,I),I=1,K14)
      WRITE(6,1001)
      DO 15 I=1,K4
   15 WRITE(6,1000)(SM1(I,J),J=1,K4)
      WRITE(6 ,1001)
      DO 16 I= 1,K4
   16 WRITE(6,1000)(SM12(I,J),J=1,K14)
      WRITE(6,1001)
      DO 17 I=1,K4
   17 WRITE(6,1000)(ALIAS(I,J),J=1,K14)
      WRITE(6,1001)
      DO 18 I=1,K4
   18 WRITE(6,1000)(MU(I,J),J=1,K14)
      WRITE(6,1001)
      DO 19 I=1,K4
   19 WRITE(6,1000)(DMR(I,J),J=1,K14)
      WRITE(6,1001)
      DO 20 I=1,K14
   20 WRITE(6,1000)(DMU(I,J),J=1,K4)
      WRITE(6,1001)
      DO 21 I=1,K14
   21 WRITE(6,1000)(Q(I,J),J=1,K14)
      WRITE(6 ,1001)
      WRITE(6,1000)(AQ(1,J),J=1,K14)
      WRITE(6,1001)
      WRITE(6,1000) B(1,1)
      GO TO 110
  300 BETSIG =BETSIG*2.
      IF(BETSIG.LT.11.0) GO TO 110
 1000 FORMAT( 10F7.2)
 1001 FORMAT(1H0)
 1002 FORMA T(1H0,F15.2)
  200 RETU RN
      END
```

```
      SUBROUTINE COMU(MU11,MU,CON)
      REAL MU11,MU
      REAL MUI,MU12,MU12MU,MU12T,MU22
      DIMENSION MU11(21,21),MU(21,35),CON(35,35)
      DIMENSION V(2),JC(21)
      DIMENSION MU12(21,35),MUI(21,21)
      DIMENSION MU12T(35,21),MU12MU(35,35)
      DIMENSION MU22(35,35)
      COMMON/A1/K,NM,KEY,KORDER
      COMMON/A5/K0,K1,K2,K3,K4,K5,K11,K14,II,KS
      COMMON/ A6/BETSIG
      AK=K
      AK1=1/(AK+2.)
      AK3=3/((AK+2.)*(AK+4.))
      AK2=1/((AK+2.)*(AK+4.))
      AK4=15/((AK+2.)*(AK+4.)*(AK+6.))
      AK5=3/((AK+2.)*(AK+4.)*(AK+6.))
      AK6=1/((AK+2.)*(AK+4.)*(AK+6.))
      II=KS-K+1
      K11=K+K
      K12=KS+1
      DO 50 I=1,K4
      DO 50 J=1,K4
   50 MU11(I,J)=0.
      DO 60 I=1,K4
      DO 60 J=1,K14
   60 MU12(I,J)=0.
      DO 70 I=1,K14
      DO 70 J=1,K14
   70 MU22(I,J)=0.
      MU11(1,1)=1.
      DO 11 I=2,K0
   11 MU11(I,I)=AK1
      IF(KORDER.EQ.4) GO TO 17
      DO 12 J=K1,K2
      MU11(J,1)=AK1
   12 MU11(1,J)=AK1
      DO 13 I=K1,K2
   13 MU11(I,I)=AK3
      DO 15 I=K1,K2
      DO 24 J=K1,K2
      IF(I.EQ.J)GO TO 24
      MU11(I,J)=AK2
   24 CONTINUE
   15 CONTINUE
      DO 16 I=K3,K4
   16 MU11(I,I)=AK2
      J=1
      DO 19 I=2,K0
      MU12(I,J)=AK3
      LL=J+1
      LLL=J+K-1
      DO 18 L=LL,LLL
   18 MU12(I,L)=AK2
      J=J+K
   19 CONTINUE
      GO TO  22
   17 DO 21 I=1,K
   21 MU12(1,I)=AK1
   22 IF(KORDER.EQ.4) GO TO 31
      DO 23 I=1,II,K
      MU22(I,I)=AK4
      JJ=I+1
      JJJ=I+K-1
      DO 23 J=JJ,JJJ
   23 MU22(J,J)=AK5
      DO 32 I=1,II,K
      JJ=I+1
      JJJ=I+K-1
      DO 32 J=JJ,JJJ
```

```
      MU22(I,J)=AK5
   32 MU22(J,I)=AK5
      IF(K.EQ.2) GO TO 33
      DO 25 I=2,KS,K
      JJ=I+1
      JJJ=I+K-2
      DO 25 J=JJ,JJJ
      MU22(I,J)=AK6
   25 MU22(J,I)=AK6
      DO 26 I=K12,K14
   26 MU22(I,I)=AK6
      GO TO 33
   31 DO 27 I=1,K
   27 MU22(I,I)=AK3
      DO 28 I=K0,K11
   28 MU22(I,I)=AK2
      DO 30 I=1,K
      DO 29 J=1,K
      IF(I.EQ.J) GO TO 29
      MU22(I,J)=AK2
   29 CONT INUE
   30 CONTIN UE
   33 V(1)=1.
      DO 14 I=1,K4
      DO 14 J=1,K4
   14 MUI(I,J)=MU11(I,J)
      CALL GJR(MUI,21,21,K4,K4,$150,JC,V)
      CALL MXMLT(MUI,MU12,MU,K4,K4,K14,21,21)
      CALL MXTRN(MU12,MU12T,K4,K14,21,35)
      CALL MXMLT(MU12T,MU,MU12MU,K14,K4,K14,35,21)
      CALL MXSUB(MU22,MU12MU,CON,K14,K14,35)
      IF(KEY.EQ.1) GO TO 200
      WRITE(6,501) K0,K1,K2,K3,K4,K5,K11,K14,II,KS
      WRITE(6,502)
      DO 400 I=1,K4
  400 WRITE(6,500) (MU11(I,J),J=1,K4)
      WRITE(6,502)
      DO 410 I=1,K4
  410 WRITE(6,500) (MU12(I,J),J=1,K14)
      WRITE( 6,502)
      DO 420 I=1,K14
  420 WRITE(6,500) (MU22(I,J),J=1,K14)
      WRITE(6,502)
      DO 430 I=1,K4
  430 WRITE(6,500)(MUI(I,J),J=1,K4)
      WRITE(6,502)
      DO 440 I=1,K4
  440 WRITE(6,500)(MU(I,J),J=1,K14)
  500 FORMAT(10F7.4)
  501 FORMAT(10I6)
  502 FORMAT(1H0)
      GO TO 200
  150 WRITE(6,160)
  160 FORMAT(T10,3HGJR)
  200 RETURN
      END


      SUBROUTINE DESCL(AUGXS,S,AVG,DSX)
      DIMENSI ON AUGXS(5,36),S(5),AVG(5),DSX(5,36)
      COMMON/ A1/K,NM,KEY,KORDER
      INTE GER U
      DO 105 I=1,K
      DO 105 U=1,NM
  105 DSX(I,U )=AUGXS(I,U)*S(I)+AVG(I)
      RETURN
      END
```

```
      SUBROUTINE FUNVAL(FX,VALUE)
      INTEGER U
      DIMENSION FX(5,36),CONST(5,10)
      COMMON /A1/K,NM,KEY,KORDER
      COMMON /A2/BMAX(5),BMIN(5),BASEN(5,36),NN,DELTA(5),AP
      COMMON/ A3/KN,NE,M
      AK=K
      ANM=NM
      DO 5 I=1,K
      DO 5 U=1,10
    5 CONST(I,U)=0.
      VALUE=0.
      KN=KN+1
      DO 109 I=1,K
      SUM= 0.
      DO 10 U=1,NM
   10 SUM= SUM+FX(I,U)
      CONS T (I,1)=SUM**2
   16 JJ=I+1
      IF(JJ.GT.K) GO TO 29
      DO 21 J=JJ,K
      SUM=0.
      DO 20 U=1,NM
   20 SUM=SUM+ FX(I,U)*FX(J,U)
      CONST( I,2)=CONST(I,2)+SUM**2
   21 CONTI NUE
   29 SUM=0.
      DO 30 U=1,NM
   30 SUM=SUM +FX(I,U)**2
      CONST(I,3)=(SUM-ANM/(AK+2.))**2
      DO 42 J=1,K
      DO 42 L=J,K
      SUM=0.
      DO 40 U=1,NM
   40 SUM=SUM+FX(I,U)*FX(J,U)*FX(L,U)
      CONST( I,4)=CONST(I,4)+SUM**2
   42 CONTIN UE
      IF(KORDER.EQ.4) GO TO 104
      DO 51 J=1,K
      IF(J.EQ.I) GO TO 51
      SUM=0.
      DO 50 U=1,NM
   50 SUM=SUM+(FX(I,U)**3)*FX(J,U)
      CONST (I,5)=CONST(I,5)+(SUM)**2
   51 CONTINUE
      DO 62 J=1,K
      IF(J.EQ.I) GO TO 62
      LL=J+1
      IF(LL.GT.K) GO TO 69
      DO 61 L=LL,K
      IF(L.EQ.I) GO TO 61
      SUM=0.
      DO 60 U=1,NM
   60 SUM=S UM+(FX(I,U)**2)*FX(J,U)*FX(L,U)
      CONST( I,6)=CONST(I,6)+SUM**2
   61 CONTI NUE
   62 CONTINUE
   69 JJ=I+1
      IF(JJ.GT.K) GO TO 79
      DO 71 J=JJ,K
      LL=J+1
      IF(LL.GT.K) GO TO 79
      DO 71 L=LL,K
      MM=L+1
      IF(MM.GT.K) GO TO 79
      DO 71  MK=MM,K
      SUM=0.
      DO 70 U=1,NM
   70 SUM=SUM+FX(I,U)*FX(J,U)*FX(L,U)*FX(MK,U)
      CONST(I,7)=CONST(I,7)+SUM**2
```

```
 71 CONTINUE
 79 JJ=I+1
    IF (JJ.GT.K) GO TO 89
    DO 81 J=JJ,K
    SUM=0 .
    DO 80 U=1,NM
 80 SUM =SUM+(FX(I,U)**2)*(FX(J,U)**2)
    CONST (I,8)=CONST(I,8)+(SUM-ANM/((AK+2)*(AK+4)))**2
 81 CONTINUE
 89 SUM=0.
    DO 90 U=1,NM
 90 SUM=SUM+FX(I,U)**4
    CONST(I,9)=(SUM-(3.*ANM)/((AK+2.)*(AK+4.)))**2
    DO 101 J=I,K
    DO 101 L=J,K
    DO 101 MK=L,K
    DO 101 IH=MK,K
    SUM=0.
    DO 100 U=1,NM
100 SUM=SUM+FX(I,U)*FX(J,U)*FX(L,U)*FX(MK,U)*FX(IH,U)
    CONST(I,10)=CONST(I,10)+SUM**2
101 CONTINUE
104 DO 105 J=KEY,KORDER
105 VALUE=VALUE+CONST(I,J)
109 CONTINUE
300 RETURN
    END


    FUNCTION IFACT(KK)
    NUM=KK
    IFACT=1
    IF(NUM.EQ.0) GO TO 7
  8 IFACT=IFACT*NUM
    NUM=NUM-1
    IF(NUM-1) 7,7,8
  7 RETURN
    END


    SUBROUTINE SCALEA(D,XS,S,AVG)
    INTEGER U
    DIMENSION D(5,36),XS(5,36),S(5),AVG(5),SSQ(5)
    COMMON/A1/K,NM,KEY,KORDER
    COMMON/A4/ R(5,2)
    DO 15 I=1,K
    AVG(I)=0.
 15 SSQ(I)=0.
    DO 30 I=1,K
    DO 25 U=1,2
 25 AVG(I)=AVG(I)+R(I,U)/2
    DO 27 U=1,2
 27 SSQ(I)=SSQ(I)+((R(I,U)-AVG(I))**2)/2
    S(I)=SSQ(I)**.5
    DO 29 U=1,NM
 29 XS(I,U)=(D(I,U)-AVG(I))/S(I)
 30 CONTINUE
    RETURN
    END
```

```
      SUBROUTINE MINMIZ(X,DEL,FXEN)
      INTEGER U
      INTEGER AP
      DIMENS ION X(5,36),DEL(5),BASEO(5,36)
      COMMON/A1/ND,NM,KEY,KORDER
      COMMON/A2/BMAX(5),BMIN(5),BASEN(5,36),NN,ADELTA(5),AP
      COMMO N/A 3/KN,NE,M,MM,DELTA(5)
      IN=1
      KM=1
      KN=0
      NE=0
      NBASE=0
C        ND=NUMBER OF DIMENSIONS
C        X=CURRENT VARIABLE VECTOR
C        BASEN=CURRENT BASE POINT
C        BASEO=LAST BASE POINT IN THE SEARCH
C        FX=FUNCTION VALUE AT X
C        FXEN= FUNCTIONAL VALUE AT CURRENT BASE POINT
C        FXBO=FUNCTIONAL VALUE AT OLD BASE POINT
  200 DO 210 I=1,ND
  210 DELTA(I)=ADELTA(I)
    1 DO 10 I=1,ND
      DO 10 U=NN,NM
C    ADD STERTING ROWS TO X
   10 X(I,U)=BASEN(I,U)
      IF(IN.EQ .1) GO TO 40
      CALL FUNVAL(X,FXBN)
C    STARTING FUNCTUON VALUE
   40 IN=IN+1
      FX = FXBN
C     EXPLORATORY MOVES
      NBASE=NBASE+1
      CALL EXPLMV(FX,X)
      IF(FX.GE.FXEN)    GO TO 3
C      SET NEW BASE POINT
    2 DO 20 I=1,ND
      DO 20 U=NN,NM
      BASEO(I,U)=BASEN(I,U)
      BASEN(I,U)=X(I,U)
   20 CONTINUE
      FXBN = FX
C       PATTERN MOVE
      DO 21 I = 1,ND
      DO 21 U=NN,NM
      X(I,U)=BASEN(I,U)*2.-BASEO(I,U)
   21 CONTINUE
      CALL FUNVAL(X,FX)
C     EXPLORATORY MOVES
      NBASE=NBASE+1
      CALL EXPLMV(FX,X)
      IF(FX.LT.FXEN) GO TO 2
C       PATTERN MOVE HAS FAILED
      NBASE=NBASE-1
      GO TO 1
    3 CONTINUE
C       CHECKING OF THE CURRENT STEP SIZE
C       IF IT IS SMALL ENOUGH STOP
C       IF IT IS LARGE, REDUCE IT TO ONE HALF AND GO BACK
      DO 30 I=1,ND
      IF(DELTA(I).GE.DEL(I))  GO TO 31
   30 CONTINUE
      GO TO 101
   31 DO 35 I=1,ND
      DELTA(I) = DELTA(I)*0.5
   35 CONTINUE
      GO TO 1
  101 RETURN
      END
```

```
      SUBROUTINE XMTX(D,X,X2)
      INTEGER U
      DIMENSION D(5,36),X(36,21),X2(36,35)
      COMM ON/ A1/K,NM,KEY,KORDER
      COMMON /A5/K0,K1,K2,K3,K4,K5,K11,K14,I1,KS
      KKK=K
      K0=K+1
      K1=K+2
      K2=(2*K)+1
      K3=(2 *K)+2
      K4=K2+IFACT(KKK)/(2*IFACT(KKK-2))
      K5=K+K4-K2
      KS=K*K
      DO 10 U=1,NM
   10 X(U,1)=1.
      DO 20 I=2,K0
      J=I-1
      DO 25 U=1,NM
   25 X(U,I )=D(J,U)
   20 CONTIN UE
      IF(KORD ER.EQ.4) GO TO 79
      J=1
      DO 30 I=K1,K2
      DO 35 U=1,NM
   35 X(U,I)=D(J,U)**2
      J=J+1
   30 CONTINUE
      IF(K.EQ.1) GO TO 49
      J=1
      M=2
      DO 40 I=K3,K4
      DO 45 U=1,NM
   45 X(U,I)=D(J,U)*D(M,U)
      M=M+1
      IF(M.GT.K) GO TO 42
      GO TO 40
   42 J=J+1
      M=J+1
   40 CONTINUE
   49 M=1
      LK=((K-1)*K)+1
      K6=K+1
      K7=K6+K
      K8=K7+K
      K9=K8+K
      K10=K9+K
      DO 51 I=1,LK,K
      DO 50 U=1,NM
   50 X2(U,I)=D(M,U)**3
      M=M+1
   51 CONT INUE
      M=1
      J=2
   63 IF(J.EQ.K6) GO TO 64
      IF (J.EQ.K7) GO TO 64
      IF(J.EQ.K8) GO TO 64
      IF(J.EQ.K9) GO TO 64
      IF(J.EQ.K10) GO TO 64
      DO 61 LJ=1,K
      IF(LJ.EQ.M) GO TO 61
      DO 60  U=1,NM
   60 X2(U,J )=D(M,U)*(D(LJ,U))**2
      J=J+1
   61 CONTINUE
      IF(J.GT.KS) GO TO 69
      GO TO 63
   64 M=M+1
      J=J+1
      GO TO 63
   66 CONTINUE
```

```
 69 KK2=K
    K12=KS+1
    K13=0
    IF(KK2-3) 72,71,71
 71 K13=IFACT(KK2)/(6*IFACT(KK2-3))
 72 K14=KS+K13
    DO 70 L=K12,K14
    DO 70 I=1,K
    JJ=I+1
    DO 70 J=JJ,K
    MM=J+1
    DO 70 M=MM,K
    DO 70 U=1,NM
 70 X2(U,L)=D(I,U)*D(J,U)*D(M,U)
    GO TO  100
 79 K4=KO
    K14=K5
    DO 80 I=1,K
    DO 85 U=1,NM
 85 X2(U,I)=D(I,U)**2
 80 CONTINU E
    IF(K.EQ.1) GO TO 100
    J=1
    M=2
    DO 90 I=KO,K5
    DO 95 U=1,NM
 95 X2(U,I)=D(J,U)*D(M,U)
    M=M+1
    IF(M.GT.K) GO TO 92
    GO TO 90
 92 J=J+1
    M=J+1
 90 CONTI NUE
100 RETUR N
    END


    SUBROUTINE EXPLMV(FX,X)
    INTEGER U
    INTEGER AP
    DIMENSI ON X(5,36)
    COMMON/ A1/ND,NM,KEY,KORDER
    COMMO N/A 2/BMAX(5),BMIN(5),BASEN(5,36),NN,ADELTA(5),AP
    COMMON/ A3/KN,NE,M,MM,DELTA(5)
    DO 201 U=NN,NM
    DO 201 I=1,ND
    X(I,U)=X(I,U)+DELTA(I)
    IF(X(I,U).GT.BMAX(I)) GO TO 180
    CALL FUNVAL(X,FXI)
    NE=KN
    IF(FXI-FX)  200,180,180
180 X(I,U)=X(I,U)-2.*DELTA(I)
    IF(X(I,U).LT.BMIN(I)) X(I,U)=X(I,U)+DELTA(I)
    CALL FUNVAL(X,FXI)
    NE=KN
    IF(FXI-FX) 200,181,181
181 X(I,U)=X(I,U)+DELTA(I)
    IF(X(I,U).GT.BMAX(I)) X(I,U)=X(I,U)-DELTA(I)
    NE=KN-2
    GO TO 202
200 FX = FXI
202 CONTINUE
201 CONTINUE
    RETURN
    END
```

APPENDIX C

DATA SETS

Tables 2 to 5 are the data sets used in this investigation. Data set A, in Table 2 is a 2^2 factorial, and is a properly designed first order experiment. It is used to see if the procedure presented behaves in a predictable way for the first order model. The addition of center points to this design should reduce bias. The region of interest is $\pm$ 1.41 for each variable.

Data set B, in Table 3 is taken from Gaylor and Merrill (8), and is also used by Dykstra (6). It is used in both first and second order model investigations for both two and three variables. The region of interest is

$$-9.69 \leq x_1 \leq 8.69$$
$$-9.90 \leq x_2 \leq 9.90$$
$$-8.52 \leq x_3 \leq 15.52$$

Data set C, in Table 4 is a central composite design with one center point and $\alpha = 2.0$. It is used for the second order model with three variables. The addition of center points should reduce bias. The region of interest is $\pm$ 2.0 for each variable.

Data set D, in Table 5 is taken from Figure 1, page 770 of Hebble and Mitchell (9). The numbers may not agree exactly with their data since they are being extracted from an illustration. The design was intended to be a central composite design with two center points. It is used with the second order model with two variables. Two regions of interest are used, one of radius 1.41 and the other of radius 1.70.

Table 2. Data Set A

ξ_1	ξ_2
-1.	1.
-1.	-1.
1.	1.
1.	-1.

Table 3. Data Set B

ξ_1	ξ_2	ξ_3
-6.389	-5.330	6.0437
-6.179	-5.549	9.0437
-4.533	-5.717	7.6283
-5.293	-6.492	7.8131
-4.004	-6.464	3.0976
-2.631	-5.320	5.4978
-3.012	-4.080	1.0688
-2.864	-4.583	4.5822
-0.979	-2.887	1.0250
-0.420	-4.094	2.2669
-1.593	-3.957	0.1869
1.338	-2.613	-0.3136
-0.787	-2.487	-2.7032
-1.649	-1.077	-2.1917
2.075	1.719	-2.2917
2.224	0.946	-2.8516
2.383	3.879	-4.2335
3.350	3.510	-4.8033
5.984	6.449	-2.1206
3.384	6.383	-2.5354

Table 4. Data Set C.

ξ_1	ξ_2	ξ_3
-1.	-1.	-1.
-1.	-1.	1.
-1.	1.	-1.
-1.	1.	1.
1.	-1.	-1.
1.	-1.	1.
1.	1.	-1.
1.	1.	1.
0.	0.	0.
-2.	0.	0.
2.	0.	0.
0.	-2.	0.
0.	2.	0.
0.	0.	-2.
0.	0.	2.

Table 5. Data Set D

ξ_1	ξ_2
1.0	1.0
0.0	1.5
0.0	0.0
1.0	-1.3
1.5	0.0
-1.4	0.0
-1.0	1.3
0.0	-1.0
-1.5	-0.8
0.5	0.3

APPENDIX D

RESULTS

Tables 6 through 37 contain the results of this investigation. For each undesigned experiment the value of uncontrollable bias, controllable bias, average bias, variance to bias ration, and average squared error, J, are calculated for five values of $\underline{\beta}_2/\sigma$. The value of $\underline{\beta}_2/\sigma$ used is referred to as B/S in the tables. The percent reduction of controllable bias, R1, percent reduction of average bias, R2, and variance do not depend on $\underline{\beta}_2/\sigma$.

The last output for each design is a summary table giving the value of the function $F(\underline{x})$, the ratio of $|(X'X)|$ of the augmented design to the $|(X'X)|$ of the original design, R1, R2, and the additional runs selected.

Table 6. Data Set A, First Order, m = 1

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
4	.1	.5	3.1	12.5	50.0
5	.2	.6	3.9	15.6	62.5
6	.2	.7	4.7	18.7	75.0
7	.2	.9	5.5	21.9	87.5
8	.3	1.0	6.2	25.0	100.0

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
4	.3	1.0	6.4	25.6	102.4	.0
5	.0	.1	.6	2.6	10.3	89.9
6	.0	.0	.1	.4	1.7	98.4
7	.1	.2	1.3	5.1	20.6	79.9
8	.1	.5	3.1	12.4	49.4	51.8

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
4	.4	1.5	9.5	38.1	152.4	.0
5	.2	.7	4.6	18.2	72.8	52.2
6	.2	.8	4.8	19.2	76.7	49.7
7	.3	1.1	6.8	27.0	108.1	29.1
8	.4	1.5	9.3	37.4	149.4	2.0

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
4	5.2	1.3	.2	.1	.0	2.0
5	9.9	2.5	.4	.1	.0	1.8
6	8.7	2.2	.3	.1	.0	1.7
7	5.8	1.4	.2	.1	.0	1.6
8	4.0	1.0	.2	.0	.0	1.5

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
4	2.4	3.5	11.5	40.1	154.4
5	2.0	2.5	6.3	20.0	74.6
6	1.9	2.4	6.5	20.8	78.3
7	1.8	2.6	8.3	28.6	109.7
8	1.9	3.0	10.8	38.8	150.9

Table 7. Data Set A, First Order, m = 2

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
4	.1	.5	3.1	12.5	50.0
6	.2	.7	4.7	18.7	75.0
8	.3	1.0	6.2	25.0	100.0

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
4	.3	1.0	6.4	25.6	102.4	.0
6	.0	.0	.1	.4	1.7	98.4
8	.1	.5	3.1	12.4	49.4	51.8

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
4	.4	1.5	9.5	38.1	152.4	.0
6	.2	.8	4.8	19.2	76.7	49.7
8	.4	1.5	9.3	37.4	149.4	2.0

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
4	5.2	1.3	.2	.1	.0	2.0
6	8.7	2.2	.3	.1	.0	1.7
8	4.0	1.0	.2	.0	.0	1.5

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
4	2.4	3.5	11.5	40.1	154.4
6	1.9	2.4	6.5	20.8	78.3
8	1.9	3.0	10.8	38.8	150.9

Table 8. Summary, Data Set A, m = 1

RUN	FUNCTION	X'X	R1	R2	X1	X2
4	2.0	1.000+00				
5	1.2	1.250+00	89.9	52.2	.000	.000
6	.5	1.500+00	98.4	49.7	.000	.000
7	.1	1.750+00	79.9	29.1	-.001	-.001
8	.0	2.000+00	51.8	2.0	.000	.000

Table 9. Summary, Data Set A, m = 2

RUN	FUNCTION	X'X	R1	R2	X1	X2
4	2.0	1.000+00				
					.000	.000
6	.5	1.500+00	98.4	49.7	.000	.000
					-.002	-.002
8	.0	2.000+00	51.8	2.0	.001	.001

Table 10. Data Set B, First Order, m = 1

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	.6	2.5	15.6	62.5	250.0
21	.7	2.6	16.4	65.6	262.5
22	.7	2.7	17.2	68.7	275.0
23	.7	2.9	18.0	71.9	287.5
24	.7	3.0	18.7	75.0	300.0
25	.8	3.1	19.5	78.1	312.5
26	.8	3.2	20.3	81.2	325.0
27	.8	3.4	21.1	84.4	337.5
28	.9	3.5	21.9	87.5	350.0
29	.9	3.6	22.7	90.6	362.5
30	.9	3.8	23.4	93.7	375.0

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
20	4.5	18.0	112.2	449.0	1795.9	.0
21	.4	1.5	9.2	36.8	147.4	91.8
22	.0	.2	1.1	4.4	17.6	99.0
23	.0	.1	.5	2.1	8.3	99.5
24	.0	.1	.7	2.7	10.8	99.4
25	.0	.1	.7	2.9	11.8	99.3
26	.0	.2	1.0	4.0	16.1	99.1
27	.1	.2	1.5	5.9	23.5	98.7
28	.1	.4	2.2	8.9	35.5	98.0
29	.1	.5	3.3	13.1	52.2	97.1
30	.2	.7	4.2	16.7	67.0	96.3

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
20	5.1	20.5	127.9	511.5	2045.9	.0
21	1.0	4.1	25.6	102.5	409.9	60.0
22	.7	2.9	18.3	73.2	292.6	85.7
23	.7	3.0	18.5	73.9	295.8	85.5
24	.8	3.1	19.4	77.7	310.5	84.8
25	.8	3.2	20.3	81.1	324.3	84.2
26	.9	3.4	21.3	85.3	341.1	83.3
27	.9	3.6	22.6	90.2	361.0	82.4
28	1.0	3.9	24.1	96.4	385.5	81.2
29	1.0	4.1	25.9	103.7	414.7	79.7
30	1.1	4.4	27.6	110.5	442.0	78.4

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
20	4.1	1.0	.2	.0	.0	21.0
21	4.2	1.1	.2	.0	.0	4.3
22	4.6	1.2	.2	.0	.0	3.4
23	3.9	1.0	.2	.0	.0	2.9
24	3.5	.9	.1	.0	.0	2.7
25	3.1	.8	.1	.0	.0	2.5
26	2.7	.7	.1	.0	.0	2.3
27	2.5	.6	.1	.0	.0	2.2
28	2.2	.6	.1	.0	.0	2.2
29	2.1	.5	.1	.0	.0	2.1
30	1.8	.5	.1	.0	.0	2.0

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	26.1	61.4	148.8	532.4	2066.9
21	5.3	8.4	29.9	106.8	414.2
22	4.1	6.3	21.7	76.5	296.0
23	3.6	5.8	21.4	76.8	298.6
24	3.5	5.8	22.2	80.4	313.6
25	3.3	5.7	22.7	83.5	326.7
26	3.1	5.7	23.6	87.6	343.4
27	3.1	5.8	24.8	92.5	363.2
28	3.1	6.0	26.3	98.5	387.6
29	3.2	6.3	28.0	105.8	416.9
30	3.1	6.5	29.7	112.5	444.0

Table 11. Data Set B, First Order, m = 2

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	.6	2.5	15.6	62.5	250.0
22	.7	2.7	17.2	68.7	275.0
24	.7	3.0	18.7	75.0	300.0
26	.8	3.2	20.3	81.2	325.0
28	.9	3.5	21.9	87.5	350.0
30	.9	3.8	23.4	93.7	375.0

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
20	4.5	18.0	112.2	449.0	1795.9	.0
22	.0	.2	1.0	4.1	16.5	99.1
24	.0	.2	1.0	3.8	15.2	99.2
26	.0	.2	1.1	4.3	17.2	99.0
28	.1	.3	2.0	8.0	32.0	98.2
30	.2	.6	3.8	15.3	61.1	96.6

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
20	5.1	20.5	127.9	511.5	2045.9	.0
22	.7	2.9	18.2	72.9	291.5	85.8
24	.8	3.2	19.7	78.8	315.2	84.6
26	.9	3.4	21.4	85.5	342.2	83.3
28	1.0	3.8	23.9	95.5	382.0	81.3
30	1.1	4.4	27.3	109.0	436.1	78.7

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
20	4.1	1.0	.2	.0	.0	21.0
22	4.0	1.0	.2	.0	.0	2.9
24	3.3	.8	.1	.0	.0	2.6
26	2.7	.7	.1	.0	.0	2.3
28	2.2	.5	.1	.0	.0	2.1
30	1.8	.4	.1	.0	.0	2.0

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	26.1	41.4	148.8	532.4	2066.9
22	3.6	5.6	21.1	75.8	294.4
24	3.4	5.8	22.3	81.4	317.8
26	3.2	5.8	23.7	87.9	344.5
28	3.0	5.9	26.0	97.6	384.0
30	3.1	6.3	29.2	111.0	438.0

Table 12. Summary, Data Set B, First Order, m = 1

RUN	FUNCTION	X'X	R1	R2	X1	X2
20	31.5	1.000+00				
21	21.6	8.217+00	91.8	80.0	-7.541	10.908
22	16.0	1.393+01	99.0	65.7	9.977	-2.310
23	13.0	1.961+01	99.5	85.5	-6.701	7.017
24	10.6	2.301+01	99.4	84.8	4.542	6.089
25	10.1	2.975+01	99.3	84.2	6.495	-6.850
26	7.9	3.641+01	99.1	83.3	-7.434	5.045
27	6.3	4.003+01	98.7	82.4	2.374	5.091
28	5.5	4.274+01	98.0	81.2	1.340	3.908
29	5.1	4.500+01	97.1	79.7	.414	3.166
30	4.9	5.064+01	96.3	78.4	-6.917	2.454

Table 13. Summary, Data Set B, First Order, m = 2

RUN	FUNCTION	X'X	R1	R2	X1	X2
20	31.5	1.000+00				
					-10.492	10.861
22	15.0	1.913+01	99.1	85.8	10.975	-5.806
					1.814	6.692
24	8.7	2.599+01	99.2	84.6	1.807	6.692
					-6.888	4.480
26	6.1	3.466+01	99.0	83.3	4.169	3.831
					6.409	-7.322
28	4.5	4.770+01	98.2	81.3	-4.181	4.790
					2.934	3.661
30	2.8	5.622+01	96.6	78.7	-5.395	3.196

Table 14. Data Set B, First Order, m = 1, 3 Variables

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	.8	3.1	19.3	77.1	308.6
21	.8	3.2	20.2	81.0	324.0
22	.8	3.4	21.2	84.9	339.4
23	.9	3.5	22.2	88.7	354.9
24	.9	3.7	23.1	92.6	370.3
25	1.0	3.9	24.1	96.4	385.7
26	1.0	4.0	25.1	100.3	401.1
27	1.0	4.2	26.0	104.1	416.6
28	1.1	4.3	27.0	108.0	432.0
29	1.1	4.5	28.0	111.9	447.4
30	1.2	4.6	28.9	115.7	462.9

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
20	.9	3.6	22.7	90.8	363.4	.0
21	3.3	13.3	82.9	331.4	1325.7	-264.8
22	3.0	12.0	74.8	299.2	1196.9	-229.4
23	2.0	8.2	51.2	205.0	819.9	-125.6
24	1.5	6.1	38.1	152.5	610.2	-67.9
25	1.3	5.2	32.2	128.8	515.2	-41.8
26	1.2	4.9	30.8	123.3	493.2	-35.7
27	1.1	4.3	27.2	108.7	434.8	-19.6
28	1.0	4.2	26.0	103.8	415.3	-14.3
29	.9	3.4	21.3	85.1	340.4	6.3
30	.8	3.1	19.6	78.4	313.6	13.7

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
20	1.7	6.7	42.0	168.0	672.0	.0
21	4.1	16.5	103.1	412.4	1649.7	-145.5
22	3.8	15.4	96.0	384.1	1536.3	-128.6
23	2.9	11.7	73.4	293.7	1174.8	-74.8
24	2.5	9.8	61.3	245.1	980.5	-45.9
25	2.3	[illegible]	56.3	225.2	900.9	-34.1
26	2.2	8.9	55.9	223.6	894.3	-33.1
27	2.1	8.5	53.2	212.8	851.3	-26.7
28	2.1	8.5	53.0	211.8	847.3	-26.1
29	2.0	7.9	49.2	197.0	787.8	-17.2
30	1.9	7.8	48.5	194.1	776.5	-15.6

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
20	16.9	4.2	.7	.2	.0	28.5
21	4.9	1.2	.2	.0	.0	20.0
22	2.3	.6	.1	.0	.0	8.7
23	1.5	.4	.1	.0	.0	4.3
24	1.7	.4	.1	.0	.0	4.1
25	1.6	.4	.1	.0	.0	3.6
26	1.5	.4	.1	.0	.0	3.2
27	1.5	.4	.1	.0	.0	3.2
28	1.4	.3	.1	.0	.0	2.9
29	1.4	.3	.1	.0	.0	2.7
30	1.4	.3	.1	.0	.0	2.7

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	30.2	35.2	70.5	196.5	703.4
21	24.1	36.5	123.1	432.4	1669.7
22	12.5	24.0	104.7	392.8	1545.0
23	7.2	16.0	77.7	298.0	1179.1
24	6.6	13.9	65.4	246.2	[illegible]
25	5.9	12.7	60.3	228.9	[illegible]
26	5.5	12.7	59.1	226.9	897.6
27	5.3	11.7	56.4	216.0	[illegible]
28	5.1	11.8	55.9	214.9	850.2
29	4.7	10.6	52.0	199.7	[illegible]
30	4.5	10.4	51.2	196.8	[illegible]

Table 15. Data Set B, First Order, m = 2, 3 Variables

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	.8	3.1	19.3	77.1	308.6
22	.8	3.4	21.2	84.9	339.4
24	.9	3.7	23.1	92.6	370.3
26	1.0	4.0	25.1	100.3	401.1
28	1.1	4.3	27.0	108.0	432.0
30	1.2	4.6	28.9	115.7	462.9

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
20	.9	3.6	22.7	90.8	363.4	.0
22	2.2	8.6	53.8	215.1	860.4	-136.8
24	1.5	5.8	36.4	145.4	581.6	-60.1
26	.9	3.6	22.3	89.0	356.1	2.0
28	.7	2.7	17.1	68.4	273.5	24.7
30	.7	2.7	17.0	68.2	272.8	24.9

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
20	1.7	6.7	42.0	168.0	672.0	.0
22	3.0	12.0	75.0	300.0	1199.8	-78.6
24	2.4	9.5	59.5	238.0	951.9	-41.7
26	1.9	7.6	47.3	189.3	757.3	-12.7
28	1.8	7.1	44.1	176.4	705.5	-5.0
30	1.8	7.4	46.0	183.9	735.7	-9.5

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
20	16.9	4.2	.7	.2	.0	28.5
22	1.7	.4	.1	.0	.0	5.1
24	1.6	.4	.1	.0	.0	3.9
26	1.6	.4	.1	.0	.0	3.1
28	1.7	.4	.1	.0	.0	2.9
30	1.4	.4	.1	.0	.0	2.6

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	30.2	35.2	70.5	196.5	700.4
22	8.1	17.1	80.1	305.1	1205.0
24	6.3	13.4	63.4	241.9	955.8
26	5.0	10.7	50.4	192.4	760.4
28	4.7	10.0	47.0	179.3	708.4
30	4.4	10.0	48.6	186.5	739.3

Table 16. Summary, Data Set B, First Order, m = 1, 3 Variables

RUN	FUNCTION	X'X	R1	R2	X1	X2	X3
20	51.1	1.000+00					
21	35.7	6.089+00	-264.8	-145.5	2.532	8.306	16.5896
22	30.4	2.337+01	-229.4	-128.6	7.172	.547	11.7875
23	22.9	8.163+01	-125.6	-74.8	-8.155	8.556	6.2400
24	20.3	9.589+01	-67.9	-45.9	1.315	3.475	9.5854
25	18.6	1.308+02	-41.8	-34.1	6.229	-1.980	8.1275
26	16.0	1.773+02	-35.7	-33.1	-6.260	5.842	3.4604
27	14.7	1.941+02	-19.6	-26.7	-.672	3.249	7.6721
28	13.9	2.429+02	-14.3	-26.1	6.061	-4.264	5.7799
29	12.0	3.162+02	6.3	-17.2	-7.225	3.427	-4.4324
30	10.7	3.452+02	13.7	-15.6	-.503	2.814	8.5830

Table 17. Summary, Data Set B, First Order, m = 2, 3 Variables

RUN	FUNCTION	X'X	R1	R2	X1	X2	X3
20	51.1	1.000+00					
					-5.895	9.988	11.5715
22	26.5	5.065+01	-136.8	-78.6	7.317	2.448	15.6330
					5.109	1.445	10.7240
24	19.9	1.060+02	-60.1	-41.7	-5.265	6.613	6.4043
					6.190	-2.276	8.8717
26	16.3	2.057+02	2.0	-12.7	-8.087	4.751	-2.4510
					.486	3.270	8.0618
28	12.8	2.506+02	24.7	-5.0	.486	3.274	8.0594
					6.789	-7.788	.4695
30	10.1	3.732+02	24.9	-9.5	-2.322	4.364	7.3622

Table 18. Data Set B, Second Order, m = 1

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	.1	.6	3.5	13.9	55.6
21	.1	.6	3.6	14.6	58.3
22	.2	.6	3.8	15.3	61.1
23	.2	.6	4.0	16.0	63.9
24	.2	.7	4.2	16.7	66.7
25	.2	.7	4.3	17.4	69.4
26	.2	.7	4.5	18.1	72.2
27	.2	.8	4.7	18.8	75.0
28	.2	.8	4.9	19.4	77.8
29	.2	.8	5.0	20.1	80.6
30	.2	.8	5.2	20.8	83.3

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
20	2.5	9.9	61.9	247.4	989.7	.0
21	.6	2.5	15.3	61.3	245.2	75.2
22	.6	2.2	13.8	55.1	220.5	77.7
23	.5	2.1	12.9	51.6	206.5	79.1
24	.5	2.1	13.2	52.9	211.4	78.6
25	.1	.6	3.5	13.8	55.3	94.4
26	.2	.6	3.8	15.0	60.2	93.9
27	.2	.7	4.7	18.6	74.5	92.5
28	.2	.9	5.8	23.1	92.4	90.7
29	.3	1.1	6.6	26.3	105.2	89.4
30	.3	1.2	7.6	30.3	121.3	87.7

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
20	2.6	10.5	65.3	261.3	1045.3	.0
21	.8	3.0	19.0	75.9	303.5	71.0
22	.7	2.8	17.6	70.4	281.6	73.1
23	.7	2.7	16.9	67.6	270.3	74.1
24	.7	2.8	17.4	69.5	278.1	73.4
25	.3	1.2	7.8	31.2	124.7	88.1
26	.3	1.3	8.3	33.1	132.4	87.3
27	.4	1.5	9.3	37.4	149.5	85.7
28	.4	1.7	10.6	42.5	170.2	83.7
29	.5	1.9	11.6	46.4	185.8	82.2
30	.5	2.0	12.8	51.1	204.6	80.4

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
20	186.9	46.7	7.5	1.9	.5	488.4
21	61.4	15.4	2.5	.6	.2	46.6
22	25.4	6.3	1.0	.3	.1	17.8
23	22.8	5.7	.9	.2	.1	15.4
24	21.9	5.5	.9	.2	.1	15.2
25	23.3	5.8	.9	.2	.1	7.3
26	16.2	4.0	.6	.2	.0	5.3
27	14.0	3.5	.6	.1	.0	5.2
28	12.0	3.0	.5	.1	.0	5.1
29	10.4	2.6	.4	.1	.0	4.8
30	9.2	2.3	.4	.1	.0	4.7

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	491.0	499.9	553.9	749.7	1535.7
21	47.4	49.6	65.6	122.5	352.1
22	19.6	20.7	35.4	88.2	299.4
23	16.1	18.1	32.3	83.0	285.7
24	15.9	18.0	32.6	84.8	293.3
25	7.6	8.5	15.1	38.5	132.0
26	5.7	6.7	13.6	38.4	137.7
27	5.6	6.7	14.6	42.6	154.7
28	3.5	6.8	15.4	47.7	175.3
29	5.3	6.7	16.5	51.3	190.6
30	5.2	6.8	17.5	55.9	209.3

Table 19. Data Set B, Second Order, m = 2

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	.1	.6	3.5	13.9	55.6
22	.2	.6	3.8	15.3	61.1
24	.2	.7	4.2	16.7	66.7
26	.2	.7	4.5	18.1	72.2
28	.2	.8	4.9	19.4	77.8
30	.2	.8	5.2	20.8	83.3

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
20	2.5	9.9	61.9	247.4	989.7	.0
22	.5	2.1	12.9	51.7	206.7	79.1
24	.5	2.1	13.0	51.8	207.4	79.0
26	.1	.6	3.6	14.4	57.5	94.2
28	.2	.8	5.0	20.1	80.4	91.9
30	.3	1.1	7.0	27.9	111.8	88.7

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
20	2.6	10.5	65.3	261.3	1045.3	.0
22	.7	2.7	16.7	66.9	267.8	74.4
24	.7	2.7	17.1	68.5	274.0	73.8
26	.3	1.3	8.1	32.4	129.7	87.6
28	.4	1.6	9.9	39.5	158.1	84.9
30	.5	2.0	12.2	48.8	195.1	81.3

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
20	186.9	46.7	7.5	1.9	.5	488.4
22	25.8	6.4	1.0	.3	.1	17.3
24	22.0	5.5	.9	.2	.1	15.0
26	21.0	5.3	.8	.2	.1	6.8
28	13.0	3.3	.5	.1	.0	5.1
30	10.0	2.5	.4	.1	.0	4.9

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	491.0	498.9	553.8	749.7	1533.7
22	17.9	19.9	34.0	84.2	285.0
24	15.7	17.8	32.2	83.6	289.1
26	7.1	8.1	14.9	39.2	136.5
28	5.5	6.7	15.0	44.7	163.3
30	5.4	6.8	17.1	53.7	200.0

Table 20. Summary, Data Set B, Second Order, m = 1

RUN	FUNCTION	X'X	R1	R2	X1	X2
20	37.6	1.000+00				
21	26.5	5.008+02	75.2	71.0	-7.771	9.345
22	19.9	1.085+04	77.7	73.1	8.950	-2.040
23	16.4	1.900+04	79.1	74.1	.177	8.162
24	15.2	2.259+04	78.6	73.4	3.465	5.633
25	14.4	1.200+05	94.4	88.1	6.165	-8.049
26	9.3	2.900+05	93.9	87.3	-9.020	4.558
27	7.7	3.353+05	92.5	85.7	-.326	5.849
28	7.0	3.777+05	90.7	83.7	4.255	3.189
29	6.1	4.689+05	89.4	82.2	-5.897	3.815
30	5.7	5.487+05	87.7	80.4	5.411	-.887

Table 21. Summary, Data Set B, Second Order, m = 2

RUN	FUNCTION	X'X	R1	R2	X1	X2
20	37.6	1.000+00				
					-8.611	9.198
22	19.6	1.366+04	79.1	74.4	9.058	-2.945
					4.183	7.272
24	13.6	3.021+04	79.0	73.8	-5.100	6.553
					6.251	-8.281
26	10.4	1.984+05	94.2	87.6	-3.090	6.460
					-8.374	2.779
28	7.3	4.455+05	91.9	84.9	4.499	4.527
					4.923	.296
30	5.8	5.862+05	88.7	81.3	-1.553	5.130

Table 22. Data Set B, Second Order, m = 1, 3 Variables

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	.2	.6	3.8	15.2	60.8
21	.2	.6	4.0	16.0	63.8
22	.2	.7	4.2	16.7	66.8
23	.2	.7	4.4	17.5	69.9
24	.2	.7	4.6	18.2	72.9
25	.2	.8	4.7	19.0	76.0
26	.2	.8	4.9	19.8	79.0
27	.2	.8	5.1	20.5	82.0
28	.2	.9	5.3	21.3	85.1
29	.2	.9	5.5	22.0	88.1
30	.2	.9	5.7	22.8	91.2

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
20	6.1	24.3	152.1	608.5	2434.0	.0
21	2.8	11.3	70.7	282.8	1131.0	53.5
22	1.7	6.7	42.0	168.2	672.8	72.4
23	.6	2.3	14.7	58.6	234.5	90.4
24	1.0	4.2	26.2	104.7	419.0	82.8
25	1.0	4.1	25.9	103.6	414.2	83.0
26	1.1	4.2	26.3	105.1	420.3	82.7
27	1.3	5.4	33.6	134.6	538.4	77.9
28	1.2	4.8	30.0	120.0	480.1	80.3
29	1.2	4.8	29.8	119.0	476.0	80.4
30	1.3	5.2	32.2	128.9	515.6	78.8

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
20	6.2	24.9	155.9	623.7	2494.8	.0
21	3.0	11.9	74.7	298.7	1194.8	52.1
22	1.8	7.4	46.2	184.9	739.6	70.4
23	.8	3.0	19.0	76.1	304.4	87.8
24	1.2	4.9	30.7	123.0	491.9	80.3
25	1.2	4.9	30.6	122.5	490.2	80.4
26	1.2	5.0	31.2	124.8	499.3	80.0
27	1.6	6.2	38.8	155.1	620.4	75.1
28	1.4	5.7	35.3	141.3	565.2	77.3
29	1.4	5.6	35.3	141.0	564.2	77.4
30	1.5	6.1	37.9	151.7	606.7	75.7

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
20	167.7	41.9	6.7	1.7	.4	1046.0
21	248.1	62.0	9.9	2.5	.6	741.1
22	167.5	41.9	6.7	1.7	.4	309.7
23	64.8	16.2	2.6	.6	.2	49.3
24	30.6	7.6	1.2	.3	.1	37.6
25	29.1	7.3	1.2	.3	.1	35.6
26	22.7	5.7	.9	.2	.1	28.3
27	16.4	4.1	.7	.2	.0	25.4
28	14.2	3.6	.6	.1	.0	20.1
29	13.9	3.5	.6	.1	.0	19.5
30	12.4	3.1	.5	.1	.0	18.9

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	1042.2	1071.0	1221.9	1660.7	3540.8
21	744.1	753.1	815.8	1030.9	1935.0
22	311.5	317.1	355.9	494.6	1049.3
23	50.0	57.3	68.3	125.4	353.7
24	38.8	42.5	68.4	108.6	529.5
25	36.5	40.5	66.2	154.7	525.8
26	29.6	33.3	59.5	154.2	527.6
27	27.0	31.6	64.7	180.5	645.8
28	21.5	25.9	55.9	161.4	555.3
29	20.9	25.2	54.4	160.6	543.7
30	20.4	24.9	56.8	170.5	625.6

Table 23. Data Set B, Second Order, m = 2, 3 Variables

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	.2	.6	3.8	15.2	60.8
22	.2	.7	4.2	16.7	66.8
24	.2	.7	4.6	18.2	72.9
26	.2	.8	4.9	19.8	79.0
28	.2	.9	5.3	21.3	85.1
30	.2	.9	5.7	22.8	91.2

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
20	6.1	24.3	152.1	608.5	2434.0	.0
22	1.6	6.3	39.5	158.1	632.2	74.0
24	.9	3.7	23.4	93.4	373.6	84.6
26	1.1	4.2	26.5	105.9	423.7	82.6
28	1.2	4.6	29.0	115.9	463.6	81.0
30	1.3	5.2	32.7	130.8	523.0	78.5

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
20	6.2	24.9	155.9	623.7	2494.8	.0
22	1.7	7.0	43.7	174.8	699.1	72.0
24	1.1	4.5	27.9	111.6	446.6	82.1
26	1.3	5.0	31.4	125.7	502.7	79.8
28	1.4	5.5	34.3	137.2	548.7	78.0
30	1.5	6.1	38.4	153.5	614.2	75.4

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
20	167.7	41.9	6.7	1.7	.4	1046.0
22	189.2	47.3	7.6	1.9	.5	330.7
24	43.6	10.9	1.7	.4	.1	48.7
26	19.9	5.0	.8	.2	.0	25.0
28	15.0	3.7	.6	.1	.0	20.5
30	12.6	3.1	.5	.1	.0	19.3

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
20	1052.2	1071.0	1201.9	1660.7	3540.8
22	332.5	337.7	374.4	505.5	1029.8
24	49.8	53.2	76.6	160.4	495.3
26	26.3	30.0	56.4	150.7	527.7
28	21.9	26.0	54.8	157.7	569.2
30	20.8	25.4	57.7	172.8	633.5

Table 24. Summary, Data Set B, m = 1, 3 Variables

RUN	FUNCTION	X'X	R1	R2	X1	X2	X3
20	56.9	1.000+00					
21	42.2	3.614+02	53.5	52.1	1.896	7.319	14.6493
22	35.5	1.223+05	72.4	70.4	7.159	.240	11.7429
23	27.0	4.149+07	90.4	87.8	-7.679	7.945	6.7095
24	23.9	1.179+08	82.8	80.3	.173	4.515	10.1653
25	21.8	2.135+08	83.0	80.4	5.939	-1.373	8.7496
26	19.2	6.881+08	82.7	80.0	-6.708	5.716	1.5400
27	17.7	1.206+09	77.9	75.1	.600	3.051	9.0807
28	16.7	4.866+09	80.3	77.3	6.305	-5.922	4.3900
29	15.1	6.522+09	80.4	77.4	-4.235	5.049	4.8983
30	14.4	8.797+09	78.8	75.7	-.778	3.128	8.0195

Table 25. Summary, Data Set B, m = 2, 3 Variables

RUN	FUNCTION	X'X	R1	R2	X1	X2	X3
20	56.9	1.000+00					
					-5.573	8.657	11.3156
22	32.2	2.001+05	74.0	72.0	6.628	2.284	14.1035
					4.686	2.245	11.3297
24	24.4	4.414+07	84.6	82.1	-7.075	6.558	3.7045
					6.947	-4.979	6.3691
26	19.4	1.203+09	82.6	79.8	-.950	5.368	9.6864
					-7.621	3.455	-4.2904
28	15.5	8.110+09	81.0	78.0	1.216	3.335	9.8507
					-1.083	4.511	7.9913
30	13.1	1.679+10	78.5	75.4	5.590	-4.439	4.6295

Table 26. Data Set C, Second Order, m = 1

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
15	.1	.5	2.8	11.4	45.6
16	.1	.5	3.0	12.2	48.6
17	.1	.5	3.2	12.9	51.7
18	.1	.5	3.4	13.7	54.7
19	.1	.6	3.6	14.4	57.7
20	.2	.6	3.8	15.2	60.8
21	.2	.6	4.0	16.0	63.8
22	.2	.7	4.2	16.7	66.8
23	.2	.7	4.4	17.5	69.9
24	.2	.7	4.6	18.2	72.9
25	.2	.8	4.7	19.0	76.0

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
15	.1	.2	1.5	5.8	23.2	.0
16	.0	.1	.7	2.7	10.8	53.6
17	.0	.0	.2	.9	3.4	85.4
18	.0	.0	.0	.1	.2	99.0
19	.0	.0	.0	.2	.6	97.3
20	.0	.0	.3	1.0	4.0	82.6
21	.0	.1	.6	2.5	10.0	56.8
22	.0	.2	1.1	4.6	18.3	21.3
23	.1	.3	2.1	8.4	33.5	-44.1
24	.1	.6	3.7	15.0	59.9	-157.5
25	.2	.9	5.6	22.4	89.8	-286.2

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
15	.2	.7	4.3	17.2	68.8	.0
16	.1	.6	3.7	14.9	59.4	13.7
17	.1	.6	3.4	13.8	55.1	20.0
18	.1	.5	3.4	13.7	54.9	20.2
19	.1	.6	3.6	14.6	58.4	15.2
20	.2	.6	4.1	16.2	64.8	5.8
21	.2	.7	4.6	18.5	73.9	-7.3
22	.2	.9	5.3	21.3	85.1	-23.7
23	.3	1.0	6.5	25.8	103.4	-50.2
24	.3	1.3	8.3	33.2	132.8	-92.9
25	.4	1.7	10.4	41.4	165.7	-140.8

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
15	49.3	12.3	2.0	.5	.1	8.5
16	50.7	12.7	2.0	.5	.1	7.5
17	51.9	13.0	2.1	.5	.1	7.2
18	50.6	12.7	2.0	.5	.1	6.9
19	46.8	11.7	1.9	.5	.1	6.8
20	41.6	10.4	1.7	.4	.1	6.7
21	36.2	9.0	1.4	.4	.1	6.7
22	31.2	7.8	1.2	.3	.1	6.6
23	25.5	6.4	1.0	.3	.1	6.5
24	19.3	4.8	.8	.2	.0	6.4
25	15.1	3.8	.6	.2	.0	6.3

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
15	8.7	9.2	12.8	25.7	77.3
16	7.7	8.1	11.2	22.6	65.9
17	7.3	7.7	10.6	21.6	62.2
18	7.1	7.5	10.4	20.7	61.[illegible]
19	7.0	7.4	10.5	21.4	65.2
20	6.9	7.4	10.8	22.6	71.6
21	6.9	7.6	11.3	25.1	80.5
22	6.8	7.5	12.0	27.9	91.8
23	6.8	7.6	13.[illegible]	32.4	110.0
24	6.7	7.7	14.7	38.6	139.2
25	6.7	7.9	16.6	47.7	172.0

Table 27. Data Set C, Second Order, m = 2

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
15	.1	.5	2.8	11.4	45.6
17	.1	.5	3.2	12.9	51.7
19	.1	.6	3.6	14.4	57.7
21	.2	.6	4.0	16.0	63.8
23	.2	.7	4.4	17.5	69.9
25	.2	.8	4.7	19.0	76.0

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
15	.1	.2	1.5	5.8	23.2	.0
17	.0	.0	.2	.9	3.4	85.4
19	.0	.0	.0	.2	.6	97.3
21	.0	.2	1.1	4.4	17.8	23.5
23	.1	.4	2.6	10.5	42.0	-80.7
25	.2	.7	4.6	18.4	73.7	-216.9

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
15	.2	.7	4.3	17.2	68.8	.0
17	.1	.6	3.4	13.8	55.1	20.0
19	.1	.6	3.6	14.6	58.4	15.2
21	.2	.8	5.1	20.4	81.6	-18.5
23	.3	1.1	7.0	28.0	111.9	-62.6
25	.4	1.5	9.4	37.4	149.6	-117.4

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
15	49.3	12.3	2.0	.5	.1	8.5
17	51.9	13.0	2.1	.5	.1	7.2
19	46.8	11.7	1.9	.5	.1	6.8
21	32.6	8.1	1.3	.3	.1	6.6
23	22.8	5.7	.9	.2	.1	6.4
25	16.0	4.0	.6	.2	.0	6.0

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
15	8.7	9.2	12.8	25.7	77.3
17	7.3	7.7	10.6	20.9	62.2
19	7.0	7.4	10.5	21.4	65.2
21	6.8	7.5	11.7	27.0	88.2
23	6.7	7.5	13.4	34.3	118.3
25	6.4	7.5	15.3	43.4	155.6

Table 28. Summary, Data Set C, m = 1

RUN	FUNCTION	X'X	R1	R2	X1	X2	X3
15	7.4	1.000+00					
16	5.7	1.778+00	53.6	13.7	.000	.000	.0004
17	4.3	2.556+00	85.4	20.0	.000	.000	.0004
18	3.2	3.333+00	99.0	20.2	-.001	-.001	-.0011
19	2.4	4.111+00	97.3	15.2	.000	.000	.0004
20	1.9	4.889+00	82.6	5.8	.000	.000	.0004
21	1.6	5.667+00	56.8	-7.3	-.001	-.001	-.0011
22	1.7	6.444+00	21.3	-23.7	.002	.002	.0020
23	2.0	7.265+00	-44.1	-50.2	.366	.366	.3661
24	2.1	9.247+00	-157.5	-92.9	-.750	-.751	-.7511
25	2.4	1.166+01	-286.2	-140.8	.758	.758	.7582

Table 29. Summary, Data Set C, m = 2

RUN	FUNCTION	X'X	R1	R2	X1	X2	X3
15	7.4	1.000+00					
					.000	.000	.0004
17	4.3	2.556+00	85.4	20.0	.000	.000	.0004
					.000	.000	.0004
19	2.4	4.111+00	97.3	15.2	-.001	-.001	-.0011
					-.392	-.392	-.3918
21	1.6	5.676+00	23.5	-18.5	.391	.391	.3911
					.716	-.636	-.6355
23	1.4	8.486+00	-80.7	-62.6	-.715	.636	.6364
					.660	.914	-.8277
25	1.3	1.495+01	-216.9	-117.4	-.661	-.915	.8270

Table 30. Data Set D, Second Order, m = 1, R = 1.41

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
10	.1	.3	1.7	6.9	27.8
11	.1	.3	1.9	7.6	30.6
12	.1	.3	2.1	8.3	33.3
13	.1	.4	2.3	9.0	36.1
14	.1	.4	2.4	9.7	38.9
15	.1	.4	2.6	10.4	41.7
16	.1	.4	2.8	11.1	44.4

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
10	.2	1.0	6.0	24.0	95.9	.0
11	.1	.6	3.7	14.8	59.3	38.1
12	.1	.4	2.3	9.4	37.6	60.8
13	.1	.2	1.4	5.7	22.8	76.2
14	.0	.1	.8	3.3	13.0	86.4
15	.0	.1	.5	1.8	7.2	92.4
16	.0	.0	.3	1.2	4.9	94.9

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
10	.3	1.2	7.7	30.9	123.6	.0
11	.2	.9	5.6	22.5	89.8	27.3
12	.2	.7	4.4	17.7	70.9	42.7
13	.1	.6	3.7	14.7	58.9	52.3
14	.1	.5	3.2	13.0	51.9	58.0
15	.1	.5	3.1	12.2	48.9	60.4
16	.1	.5	3.1	12.3	49.3	60.1

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
10	13.9	3.5	.6	.1	.0	4.3
11	16.6	4.2	.7	.2	.0	3.7
12	19.3	4.8	.8	.2	.0	3.4
13	21.9	5.5	.9	.2	.1	3.2
14	23.9	6.0	1.0	.2	.1	3.1
15	24.5	6.1	1.0	.2	.1	3.0
16	23.7	5.9	.9	.2	.1	2.9

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
10	4.6	5.5	12.0	35.2	127.9
11	4.0	4.6	9.3	26.2	93.6
12	3.6	4.1	7.9	21.1	74.3
13	3.4	3.8	6.9	18.0	62.2
14	3.2	3.6	6.3	16.1	55.0
15	3.1	3.5	6.1	15.2	51.9
16	3.0	3.4	6.0	15.3	52.3

Table 31. Data Set D, Second Order, m = 2, R = 1.41

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
10	.1	.3	1.7	6.9	27.8
12	.1	.3	2.1	8.3	33.3
14	.1	.4	2.4	9.7	38.9
16	.1	.4	2.8	11.1	44.4

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
10	.2	1.0	6.0	24.0	95.9	.0
12	.1	.4	2.4	9.4	37.7	60.7
14	.0	.1	.8	3.3	13.0	86.4
16	.0	.0	.3	1.2	4.9	94.9

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
10	.3	1.2	7.7	30.9	123.6	.0
12	.2	.7	4.4	17.8	71.0	42.6
14	.1	.5	3.2	13.0	51.9	58.0
16	.1	.5	3.1	12.3	49.3	60.1

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
10	13.9	3.5	.6	.1	.0	4.3
12	19.3	4.8	.8	.2	.0	3.4
14	23.9	6.0	1.0	.2	.1	3.1
16	23.7	5.9	.9	.2	.1	2.9

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
10	4.6	5.5	12.0	35.2	127.9
12	3.6	4.1	7.9	21.2	74.4
14	3.2	3.6	6.3	16.1	55.0
16	3.1	3.4	6.0	15.3	52.3

Table 32. Summary, Data Set D, m = 1, R = 1.41

RUN	FUNCTION	X'X	R1	R2	X1	X2
10	28.2	1.000+00				
11	24.9	1.445+00	38.1	27.3	-.024	-.342
12	21.9	1.904+00	60.8	42.7	-.020	-.265
13	19.3	2.376+00	76.2	52.3	-.016	-.198
14	17.0	2.858+00	86.4	58.0	-.013	-.141
15	15.1	3.350+00	92.4	60.4	-.010	-.090
16	13.5	3.850+00	94.9	60.1	-.007	-.049

Table 33. Summary, Data Set D, m = 2, R = 1.41

RUN	FUNCTION	X'X	R1	R2	X1	X2
10	28.2	1.000+00				
					-.021	-.289
12	21.9	1.908+00	60.7	42.6	-.021	-.289
					-.015	-.173
14	17.0	2.858+00	86.4	58.0	-.018	-.173
					-.009	-.077
16	13.5	3.842+00	94.9	60.1	-.010	-.082

Table 34. Data Set D, Second Order, m = 1, R = 1.7

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
10	.1	.3	1.7	6.9	27.8
11	.1	.3	1.9	7.6	30.6
12	.1	.3	2.1	8.3	33.3
13	.1	.4	2.3	9.0	36.1
14	.1	.4	2.4	9.7	38.9
15	.1	.4	2.6	10.4	41.7
16	.1	.4	2.8	11.1	44.4

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
10	.0	.1	.5	1.9	7.6	.0
11	.0	.0	.3	1.2	4.9	35.1
12	.0	.1	.4	1.8	7.2	5.5
13	.0	.1	.8	3.1	12.5	-64.5
14	.1	.2	1.3	5.0	20.2	-166.5
15	.1	.3	2.1	8.4	33.5	-342.1
16	.1	.5	3.2	12.7	50.9	-571.2

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
10	.1	.4	2.2	8.8	35.4	.0
11	.1	.4	2.2	8.9	35.5	-.3
12	.1	.4	2.5	10.1	40.5	-14.5
13	.1	.5	3.0	12.1	48.6	-37.4
14	.1	.6	3.7	14.8	59.1	-67.1
15	.2	.8	4.7	18.8	75.2	-112.6
16	.2	1.0	6.0	23.8	95.3	-169.6

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
10	64.5	16.1	2.6	.6	.2	5.7
11	57.5	14.4	2.3	.6	.1	5.1
12	46.2	11.6	1.8	.5	.1	4.7
13	36.7	9.2	1.5	.4	.1	4.5
14	29.3	7.3	1.2	.3	.1	4.3
15	21.6	5.4	.9	.2	.1	4.1
16	16.3	4.1	.7	.2	.0	3.9

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
10	5.8	6.1	7.9	14.5	41.1
11	5.2	5.5	7.3	14.0	40.6
12	4.8	5.1	7.2	14.8	45.2
13	4.6	4.9	7.5	16.6	53.0
14	4.5	4.9	8.0	19.1	63.4
15	4.2	4.8	8.8	22.9	79.2
16	4.1	4.8	9.3	27.7	99.2

Table 35. Data Set D, Second Order, m = 2, R = 1.7

VALUE OF UNCONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0
10	.1	.3	1.7	6.9	27.8
12	.1	.3	2.1	8.3	33.3
14	.1	.4	2.4	9.7	38.9
16	.1	.4	2.8	11.1	44.4

VALUE AND PERCENT REDUCTION OF CONTROLLABLE PORTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R1
10	.0	.1	.5	1.9	7.6	.0
12	.0	.1	.4	1.6	6.3	17.5
14	.1	.2	1.6	6.2	24.8	-227.6
16	.1	.5	2.9	11.7	46.9	-518.9

VALUE AND PERCENT REDUCTION OF BIAS

RUN	B/S=0.5	1.0	2.5	5.0	10.0	R2
10	.1	.4	2.2	8.8	35.4	.0
12	.1	.4	2.5	9.9	39.6	-12.0
14	.2	.6	4.0	15.9	63.7	-80.2
16	.2	.9	5.7	22.8	91.4	-158.4

VARIANCE TO BIAS RATIO, VALUE OF VARIANCE

RUN	B/S=0.5	1.0	2.5	5.0	10.0	VARIANCE
10	64.5	16.1	2.6	.6	.2	5.7
12	45.7	11.4	1.8	.5	.1	4.5
14	26.3	6.6	1.1	.3	.1	4.2
16	16.9	4.2	.7	.2	.0	3.9

VALUE OF J

RUN	B/S=0.5	1.0	2.5	5.0	10.0
10	5.6	6.1	7.9	14.5	41.1
12	4.6	4.9	7.0	14.4	44.1
14	4.4	4.8	8.2	20.1	67.9
16	4.1	4.8	9.6	26.7	95.2

Table 36. Summary, Data Set D, m = 1, R = 1.7

RUN	FUNCTION	X'X	R1	R2	X1	X2
10	3.2	1.000+00				
11	1.9	1.605+00	35.1	-.3	-.074	-1.110
12	1.3	2.357+00	5.5	-14.5	-.016	.007
13	1.0	3.107+00	-64.5	-37.4	-.009	.003
14	.9	3.856+00	-166.5	-67.1	-.002	-.001
15	1.1	5.244+00	-342.1	-112.6	-.077	-1.106
16	.9	6.897+00	-571.2	-169.6	.837	.799

Table 37. Summary, Data Set D, m = 2, R = 1.7

RUN	FUNCTION	X'X	R1	R2	X1	X2
10	3.2	1.000+00				
					-.089	-1.259
12	1.2	2.698+00	17.5	-12.0	-.006	.089
					.401	.411
14	.7	4.240+00	-227.6	-80.2	-.302	-.421
					-.660	.670
16	.7	6.863+00	-518.9	-158.4	.698	-.912

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